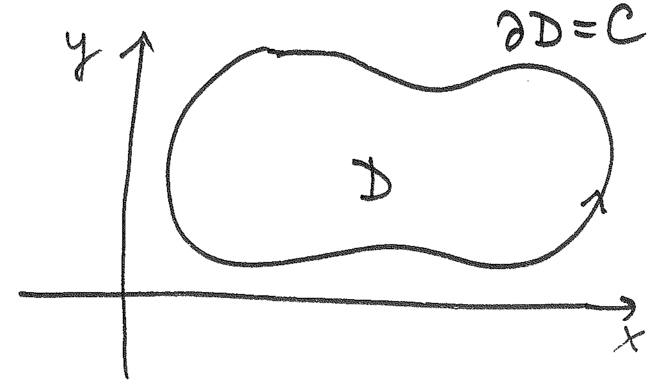
Lecture 44

16.4 GREEN's Theorem

Green's Thm gives relation between line integral along the boundary of a region and double integral over this region.



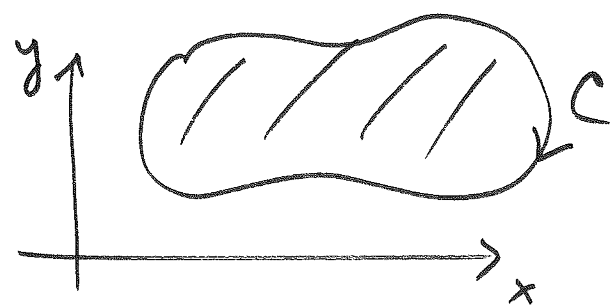
or

line integral along a simple closed contour C and double integral over domain bounded by C.

no self intersections

Positive orientation:

as one moves along C, domain D always stays on the left



Negative orientation

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Green's Thm

$\vec{F} = P\hat{i} + Q\hat{j}$

Let C be a positively oriented, piecewise smooth, simple closed curve in plane and let D be the region bounded by C. If P and Q have continuous partial derivatives, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Recall, we can write

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy \qquad \vec{F} = P\hat{i} + Q\hat{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b \left(P\hat{i} + Q\hat{j}\right)\cdot\left(x'\hat{i} + y'\hat{j}\right)dt =$$

$$= \int_a^b P\underbrace{x'\,dt}_{dx} + Q\underbrace{y'\,dt}_{dy} = \int_C P\,dx + Q\,dy$$

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Notation

$$\int_C P\,dx + Q\,dy \qquad \oint_C P\,dx + Q\,dy \qquad \oint_C P\,dx + Q\,dy$$

$$\oint_{\partial D} P\,dx + Q\,dy$$

$$\therefore \boxed{\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Green's Thm is a counterpart of the Fundamental Thm for double integrals.

Compare

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy \quad \text{with} \quad \int_a^b F'(x)\,dx = F(b) - F(a)$$

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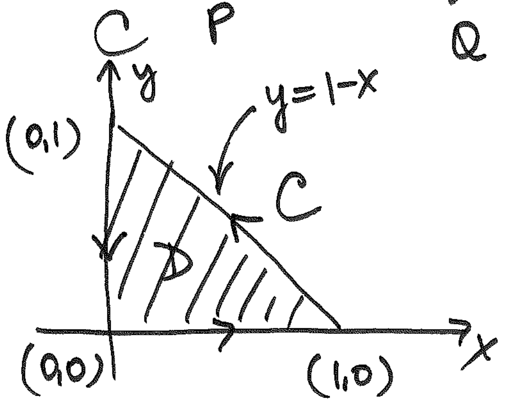
<u>Ex</u> Evaluate

$$\int_C \underbrace{x^4}_{P}\,dx + x\underbrace{y}_{Q}\,dy \overset{\text{Green's}}{\underset{\text{Thm}}{=}}$$



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^4)\right] dA =$$

$$= \iint_D (y - 0)\,dA = \iint_D y\,dA =$$

D as domain of type I: $D = \{(x,y):\ 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx = \int_{x=0}^{1} \frac{y^2}{2}\Big|_{y=0}^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx =$$

$$= \frac{1}{2}(-1)\frac{(1-x)^3}{3}\Big|_0^1 = \boxed{\frac{1}{6}}$$

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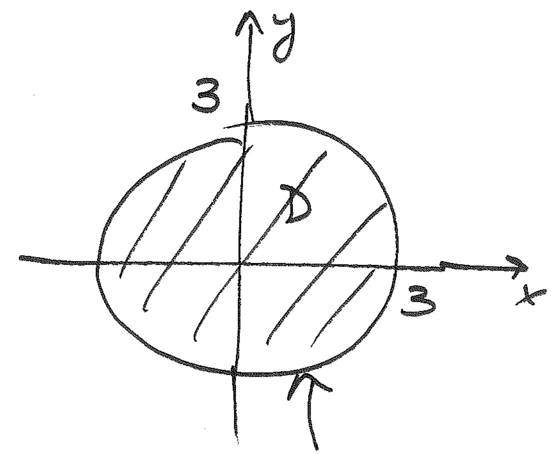
Ex Evaluate

$$\int_C \underbrace{(3y - e^{\sin x})}_{P}\,dx + \underbrace{(7x + \sqrt{y^4+1})}_{Q}\,dy =$$

where C is the circle $x^2+y^2=9$



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x+\sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right)\right] dA =$$

$D = \{(r,\theta): 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

$$= \iint_D (7-3)\,dA = 4 \iint_D dA \underset{\text{coordinates}}{\overset{\text{polar}}{=}} 4\int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\,dr\,d\theta =$$

$$= 4\int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{3} r\,dr = 4\cdot 2\pi \cdot \frac{r^2}{2}\Big|_0^3 = 4\cdot 2\pi\cdot\frac{9}{2} = 36\pi$$

or $4\cdot A(D) =$

$= 4\cdot \pi r^2\big|_{r=3} = 36\pi$

Note $\iint_D dA = A(D)$: area of D

Sometimes it is easier to evaluate line integral $\oint_C$, sometimes – double $\iint_D$, so Green's Thm can be used in both directions.

Ex if $P(x,y) = Q(x,y) = 0$ on C, then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \overset{\text{Green's}}{\underset{\text{Thm}}{=}} \oint_C \underbrace{P}_{=0} dx + \underbrace{Q}_{=0} dy = 0$$

no matter what values P and Q assume inside D.

Note Green's Thm can be used to compute areas of domains

Recall

$$A(D) = \iint_D 1 \cdot dA. \qquad \text{Choose } P \text{ and } Q: \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

e.g. let $P(x,y) = 0$, $Q(x,y) = x$ or $P(x,y) = -y$, $Q(x,y) = 0$ or $P(x,y) = -\frac{1}{2}y$, $Q(x,y) = \frac{1}{2}x$

Then

$$A(D) = \iint_D 1 \cdot dA = \oint_C x\,dy = -\oint_C y\,dx = \oint_C \left(-\tfrac{1}{2}y\right)dx + \tfrac{1}{2}x\,dy$$

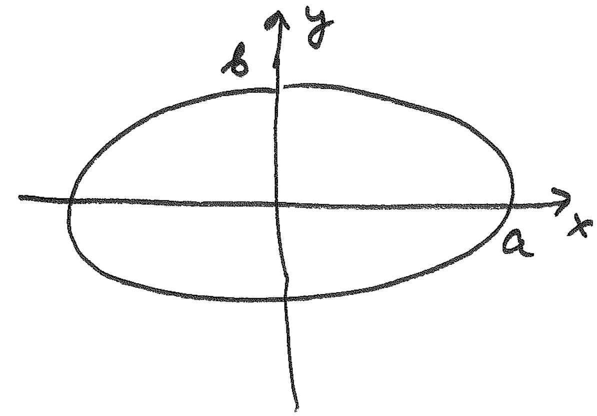
$P=0,\ Q=x$ $\quad$ $P=-y,\ Q=0$

$$= \frac{1}{2}\oint_C x\,dy - y\,dx$$

$P=-\frac{1}{2}y,\ Q=\frac{1}{2}x$

Ex Find the area enclosed by ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Parametric eq^{ns} of ellipse:

$$x = a\cos t \qquad 0 \le t \le 2\pi$$
$$y = b\sin t$$

Use $P = -\frac{1}{2}y,\ Q = \frac{1}{2}x$

$$A(D) = \frac{1}{2}\oint_C x\,\underbrace{dy}_{y'(t)\,dt} - y\,\underbrace{dx}_{x'(t)\,dt} = \frac{1}{2}\int_0^{2\pi} \underbrace{a\cos t}_{x} \cdot \underbrace{(b\cos t)}_{y'}\,dt - \underbrace{b\sin t}_{y}\,\underbrace{(-a\sin t)}_{x'}\,dt =$$

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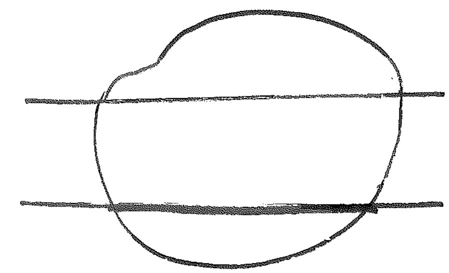
$$= \frac{1}{2}\int_0^{2\pi} ab\,(\underbrace{\cos^2 t + \sin^2 t}_{=1})\,dt = \frac{1}{2}ab\int_0^{2\pi} dt = \frac{1}{2}ab \cdot 2\pi = \pi ab$$

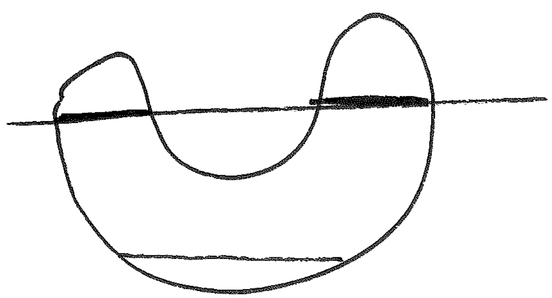
simple region



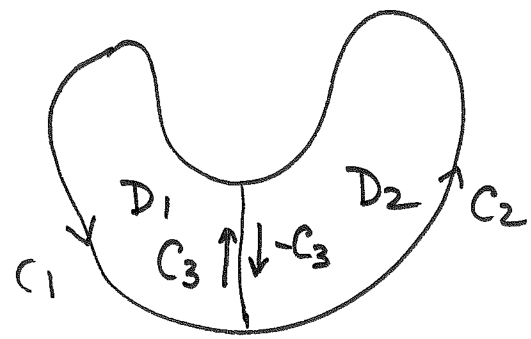
intersection w/ horizontal line is in one segment

Extended Versions of Green's Thm

D is union of simple regions



not simple region



$D = D_1 \cup D_2$

D_1, D_2: simple regions

$\partial D_1 = C_1 \cup C_3$

$\partial D_2 = C_2 \cup (-C_3)$

$$\oint_{C_1 \cup C_3} P\,dx + Q\,dy \overset{\text{Green's}}{\underset{\text{Thm}}{=}} \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$+$

$$\oint_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint_{D_1} + \iint_{D_2} = \iint_{D}$$

$$\int_{C_3} + \int_{-C_3} = 0$$ (cancel because of opposite direction)

$$\therefore \oint_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Ex Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane bounded by circles $x^2+y^2=1$ and $x^2+y^2=4$.