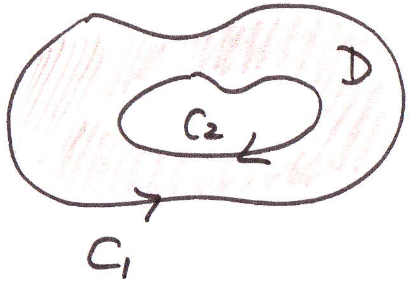
Lecture 45

GREEN's Thm can be extended to regions with holes (not simply-connected regions).



$\partial D = C_1 \cup C_2$

Note orientation of C_1 & C_2: domain D always stays on the left as ∂D is traversed.

C_1: counterclockwise

C_2: clockwise

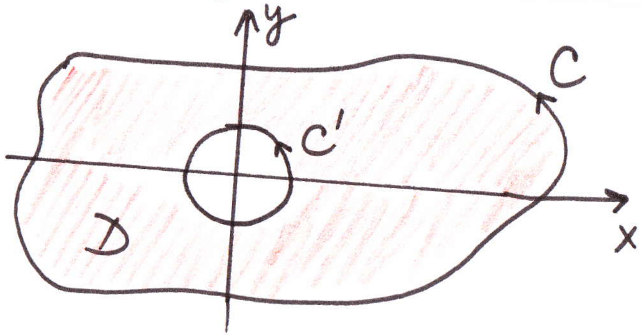
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + $$

$$+ \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{\text{Green's}}{\underset{\text{Thm}}{=}} \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

Line integrals along common boundaries are in opposite directions $\rightarrow$ cancel

$$\therefore \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_{C=\partial D} P\,dx + Q\,dy$$

Ex If $\vec{F}(x,y) = \dfrac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}$, show $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.



C: arbitrary positively oriented simple closed path that encloses origin

C': circle of rad a

$\partial D = C \cup (-C')$: positively oriented

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy \overset{\text{Green's}}{\underset{\text{Thm}}{=}} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$= \iint_D \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right] dA = \iint_D 0 \cdot dA = 0$$

$$P = \frac{-y}{x^2+y^2};\quad Q = \frac{x}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2-2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-1\cdot(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$$

C': $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$ $\Rightarrow$ $\vec{r}(t) = a\cos t \cdot \hat{i} + a\sin t \cdot \hat{j}$

$x^2 + y^2 = a^2$ $\qquad$ $\vec{r}\,'(t) = -a\sin t \cdot \hat{i} + a\cos t \cdot \hat{j}$

$$\therefore \int_C \vec{F} . d\vec{r} = \int_{C'} \vec{F} . d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt =$$

$$= \int_0^{2\pi} \frac{-a\sin t \cdot \hat{i} + a\cos t \cdot \hat{j}}{a^2} \cdot \left(-a\sin t \cdot \hat{i} + a\cos t \cdot \hat{j}\right) dt =$$

$$= \int_0^{2\pi} \frac{a^2\sin^2 t + a^2\cos^2 t}{a^2}\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

16.5 Curl and Divergence (Bonus)

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$: vector field

$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$: gradient differential operator

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

curl of $\vec{F}$

Ex $\vec{F}(x,y,z) = xz\cdot\hat{i} + xyz\cdot\hat{j} - y^2\hat{k}$. Find curl $\vec{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]$$

$$-\hat{j}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right) + \hat{k}\left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right) = \hat{i}\left[-2y - xy\right] - \hat{j}(-x) + \hat{k}[yz - 0]$$

$$= -y(2+x)\hat{i} + x\hat{j} + yz\hat{k}$$

Thm If $f=f(x,y,z)$ has continuous 2^{nd} order partial derivatives, then

$$\operatorname{curl}(\nabla f)=0$$

□

$$\operatorname{curl}(\nabla f)=\nabla\times(\nabla f)=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{vmatrix}=\hat{\imath}\left(\cancel{\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}}^{\,0}\right)$$

$$-\hat{\jmath}\left(\cancel{\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 f}{\partial z\partial x}}^{\,0}\right)+\hat{k}\left(\cancel{\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}}^{\,0}\right)=0\hat{\imath}+0\hat{\jmath}+0\cdot\vec{k}=\vec{0}$$

If $\vec{F}$ is conservative, i.e. $\exists f$: $\vec{F}=\nabla f \Rightarrow \operatorname{curl}\vec{F}=0$

Thm If $\vec{F}$ is a vector field in $\mathbb{R}^3$ whose components have continuous partial derivatives and $\operatorname{curl}\vec{F}=0$, then $\vec{F}$ is conservative vector field.

Divergence

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$: vector field

$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$: divergence of $\vec{F}$

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \Rightarrow \boxed{\text{div}\,\vec{F} = \nabla\cdot\vec{F}}$

$\Rightarrow \text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ ✓

Thm If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in $\mathbb{R}^3$, then

$$\underbrace{\text{div curl}\,\vec{F}}_{\nabla\cdot(\nabla\times\vec{F})} = 0$$

$\text{div}(\nabla f) = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$: Laplacian of f

$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$: Laplace operator

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$: Laplace's eq^{n}

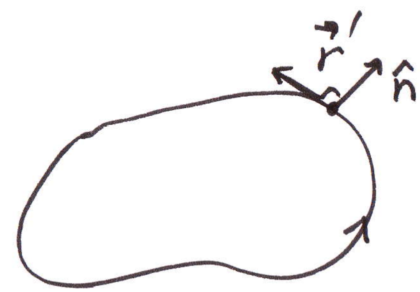
Vector forms of Green's Thm

$\vec{F} = P\vec{i} + Q\vec{j}$: vector field

$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy$: line $\int$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

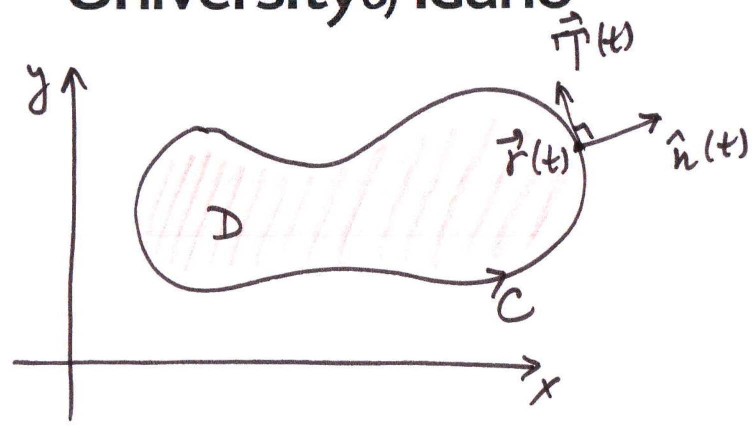
$$\Rightarrow \quad (\operatorname{curl}\vec{F}) \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underbrace{\hat{k} \cdot \hat{k}}_{=1} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$



$$\oint_C \vec{F} \cdot d\vec{r} \overset{\text{Green's}}{\underset{\text{Thm}}{=}} \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\operatorname{curl}\vec{F} \cdot \hat{k}} dA$$

$$\therefore \quad \boxed{\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{tangential component of } \vec{F}} = \iint_D (\operatorname{curl}\vec{F})\,\vec{k}\, dA}$$

line $\int$ of tangential component of $\vec{F}$ as double $\int$ of the vertical component of curl F



$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} \qquad a \le t \le b$$

$$\vec{T}(t) = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{x'(t)}{|\vec{r}\,'(t)|}\,\vec{i} + \frac{y'(t)}{|\vec{r}\,'(t)|}\,\vec{j}$$

$$\hat{n}(t) = \frac{y'(t)}{|\vec{r}\,'(t)|}\,\vec{i} - \frac{x'(t)}{|\vec{r}\,'(t)|}\,\vec{j}$$: outward unit normal

Recall

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\,\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}_{|\vec{r}\,'(t)|}\,dt$$

Then

$$\oint_C \underbrace{\vec{F}\cdot\hat{n}}_{\text{normal component of } \vec{F}}\,ds = \int_a^b (\vec{F}\cdot\hat{n})(t)\cdot|\vec{r}\,'(t)|\,dt = \int_a^b \left(\frac{P\cdot y'}{|\vec{r}\,'|} - \frac{Q\cdot x'}{|\vec{r}\,'|}\right)|\vec{r}\,'|\,dt =$$

$$= \int_a^b P y'\,dt - Q\,x'\,dt = \int_C P\,dy - Q\,dx \overset{\text{Green's Thm}}{=} \iint_D \left(\frac{\partial}{\partial x}(P) - \frac{\partial}{\partial y}(-Q)\right) dA$$

$P \to -Q$

$Q \to P$

$$= \iint_D \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}_{\operatorname{div} \vec{F}} dA$$

$$\Rightarrow \boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \operatorname{div} \vec{F}(x,y)\,dA}$$

<u>divergence theorem</u>

Final Exam Review

Cumulative

$\approx$ 4 problems on new material

- cylindrical, spherical coordinates
- triple integrals
- change of variables
- line integrals (line $\int$ wrt arclength, wrt x and wrt y)
- conservative vector fields, finding potential function f: $\nabla f = \vec{F}$
- independence of path
- Green's thm

Bonus: curl F, div F, divergence thm

$\approx$ 3 problems on previous material

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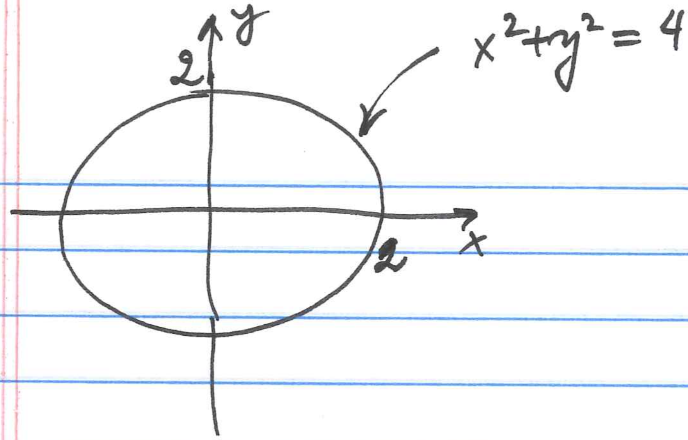
Note sheet: 2 sided (double sided) sheet : 8.5×11 in^2

Ex Evaluate line integrals by two methods: (a) directly and (b) using Green's Thm.

$$\oint_C (x-y)\,dx + (x+y)\,dy$$

where C is the circle with center at origin and radius 2.



$x = 2\cos t$
$y = 2\sin t$
$0 \le t \le 2\pi$

Recall

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(x(t),y(t))\cdot x'(t)\,dt \; + $$

curve C: $x = x(t)$,
$y = y(t)$
$a \le t \le b$

$$+ \; Q(x(t),y(t))\cdot y'(t)\,dt$$

In our case, $P(x,y) = x-y$, $Q(x,y) = x+y$

C: $x = 2\cos t$, $y = 2\sin t$, $a=0$, $b=2\pi$

$$\Rightarrow \oint_C (x-y)\,dx + (x+y)\,dy =$$

$$= \int_0^{2\pi} (2\cos t - 2\sin t)\underbrace{(-2\sin t)}_{x'(t)}dt \; +$$

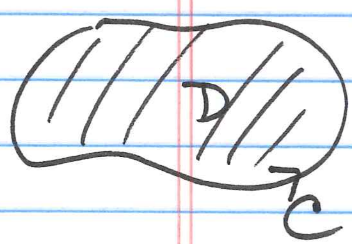
$$+ (2\cos t + 2\sin t)\cdot \underbrace{(2\cos t)}_{y'(t)}dt \; =$$

$$= 4 \int_0^{2\pi} \left(-\cancel{\cos t \sin t} + \underbrace{\sin^2 t + \cos^2 t}_{1} + \cancel{\sin t \cdot \cos t} \right) dt$$

$$= 4 \int_0^{2\pi} dt = \boxed{8\pi}$$

(b) Green's Thm

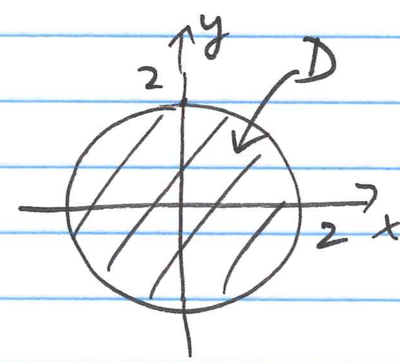
$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



$P = x - y$ $\qquad$ $Q = x + y$

$\frac{\partial Q}{\partial x} = 1$ $\qquad$ $\frac{\partial P}{\partial y} = -1$

$$\therefore \oint_C (x-y)\,dx + (x+y)\,dy =$$



$$= \iint_{\substack{D \\ x^2+y^2 \le 4}} \left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y) \right] dA =$$

$$= \iint_D \left(1 - (-1) \right) dA = 2 \iint_D dA = 2 \cdot \text{area of } D =$$

$$= 2 \cdot \pi \cdot 2^2 = \boxed{8\pi}$$

Ex Evaluate $\int_C (x+y)\, ds$ — line $\int$ wrt arclength

C: same as before

$x = 2\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

Recall

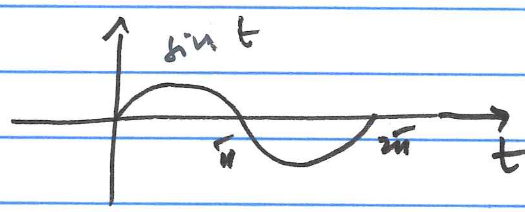
$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

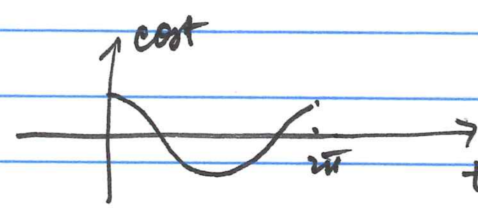
$C: \quad x = x(t), \quad y = y(t), \quad a \le t \le b$

$\Rightarrow$

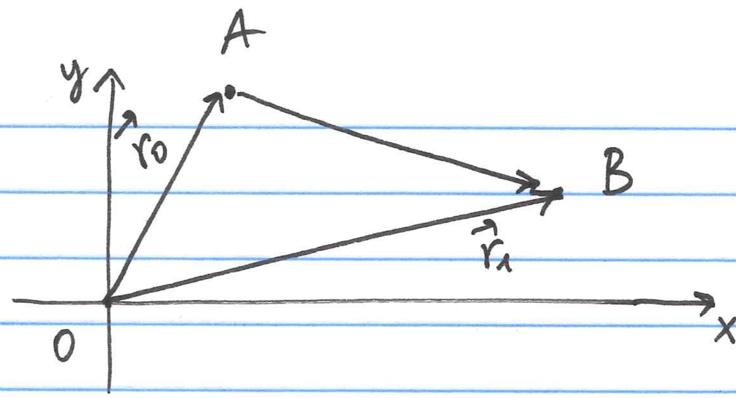
$$\int_C (x+y)\, ds = \int_0^{2\pi} (2\cos t + 2\sin t) \underbrace{\sqrt{(-2\sin t)^2 + (2\cos t)^2}}_{\sqrt{4} = 2}\, dt$$

$$= 4 \int_0^{2\pi} (\cos t + \sin t)\, dt = 0$$



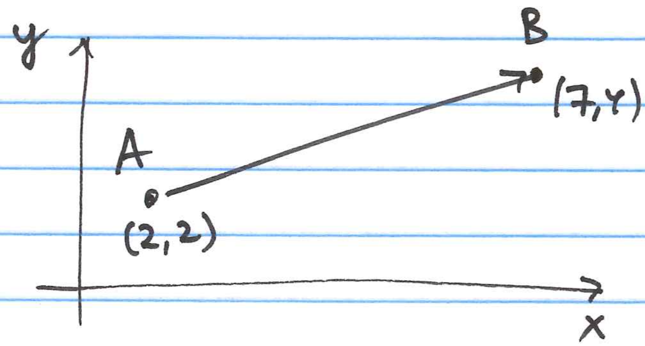


Ex



line segment AB:

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$$



Find vector eq^{n} of segment $\vec{AB}$.

$\vec{r_0} = \langle 2,2 \rangle \qquad \vec{r_1} = \langle 7,4 \rangle$

$$\vec{r}(t) = (1-t)\vec{r_0} + t \cdot \vec{r_1} = (1-t) \cdot \langle 2,2 \rangle + t \langle 7,4 \rangle =$$

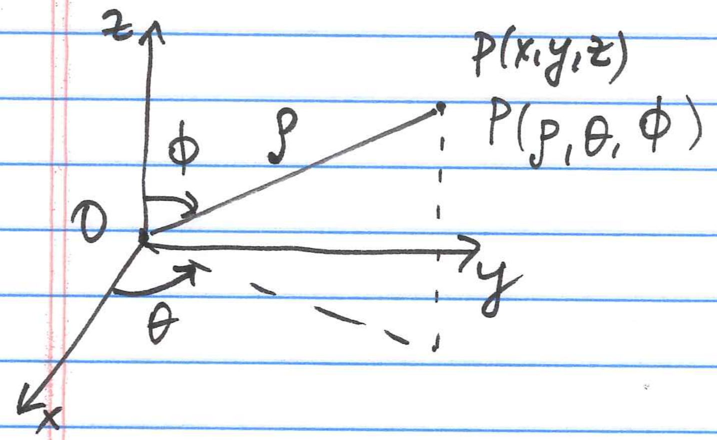
$$= \langle 2(1-t) + 7t,\ 2(1-t) + 4t \rangle =$$

$$= \langle 5t+2,\ 2t+2 \rangle \qquad 0 \le t \le 1$$

or $\quad x = 5t+2, \quad y = 2t+2$

Review problems on work w/ variable force acting along curve C.

<u>Ex</u> Point $(0, \sqrt{3}, 1)$ in rectangular coordinates. Convert to spherical.



$$x = \rho \cos\theta \sin\phi$$

$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\phi$$

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

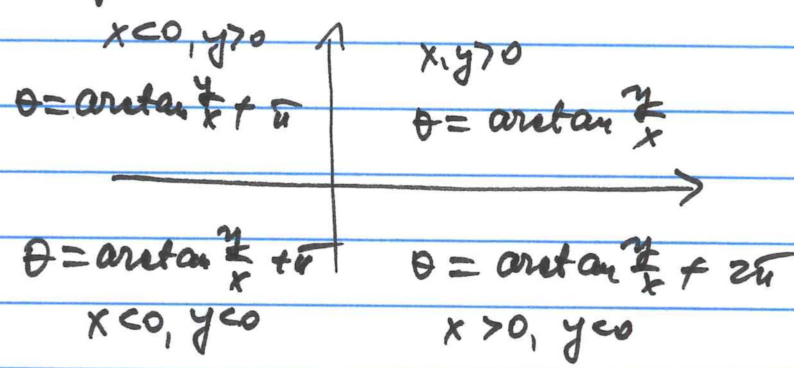
$x = 0, \; y = \sqrt{3}, \; z = 1$

$$\rho^2 = x^2 + y^2 + z^2 \Rightarrow \rho = \sqrt{x^2+y^2+z^2} =$$

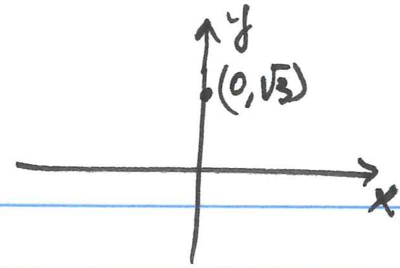
$$= \sqrt{0^2 + (\sqrt{3})^2 + 1^2} = 2$$

$$z = \rho\cos\phi \Rightarrow \cos\phi = \frac{z}{\rho} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$$

$$\tan\theta = \frac{y}{x}$$



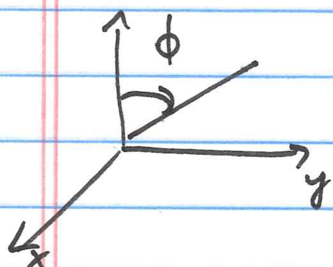
$x=0,\ y=\sqrt{3}$

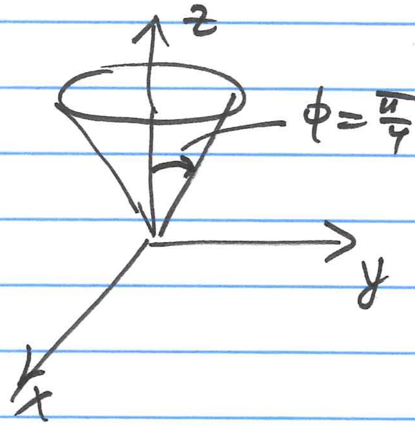


$\theta=\frac{\pi}{2}$

Ex Identify surface: $\phi=\frac{\pi}{4}$

It is a cone, $z>0$





Review integrals in cylindrical & spherical coordinates, i.e. triple integrals

– Max-min problems $f(x,y)$

– crit. points (x_0,y_0) $f_x=f_y=0$

– 2^{nd} derivative test – to classify

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=f_{xx}\cdot f_{yy}-(f_{xy})^2$$

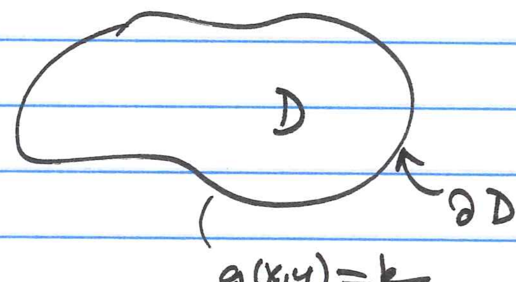
if $D>0$, $f_{xx}>0$ at (x_0,y_0) $\Rightarrow$ (x_0,y_0) is a pt of local min if $D>0$, $f_{xx}<0$ $\Rightarrow$ (x_0,y_0) is a pt of local max

if $D<0$ $\Rightarrow$ (x_0,y_0) is a saddle pt

if $D=0$: test is inconclusive
(use def of local min/max to analyze)

— method of Lagrange multipliers is used to find max/min on the boundary

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=k \end{cases}$$



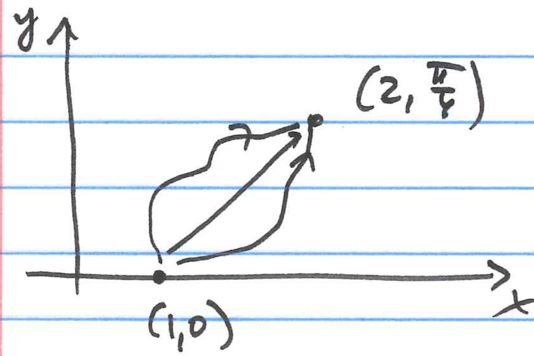
or
$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y)=k \end{cases}$$

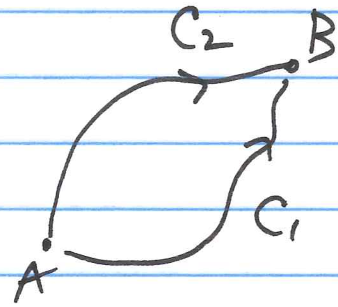
__Ex__ Show that the line integral is independent of path and evaluate the integral.

S16.3

$$\int_C \tan y \, dx + x \sec^2 y \, dy$$

C is any path from $(1,0)$ to $(2,\frac{\pi}{4})$





In general, $\displaystyle\int_{C_1} P\,dx + Q\,dy \neq \int_{C_2} P\,dx + Q\,dy$

Recall

$$\int_C \vec{F}\,d\vec{r} = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

$\vec{F}$: vector field

$\vec{F} = P\hat{\imath} + Q\hat{\jmath}$

$$\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

$$C: \quad \vec{r} = \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

We know that $\displaystyle\int_C \vec{F}\,d\vec{r}$ is independent of path only if $\vec{F}$ is conservative, i.e. there exists a potential function f:

$$\nabla f = \vec{F}$$

$\langle f_x, f_y \rangle = \langle P, Q \rangle$

$f_{xy} = P_y = \dfrac{\partial P}{\partial y}$; $\quad f_{yx} = Q_x = \dfrac{\partial Q}{\partial x}$

Vector field $\vec{F}$ is conservative if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$P(x,y) = \tan y \qquad Q(x,y) = x\sec^2 y$

$\dfrac{\partial P}{\partial y} = \sec^2 y \qquad \dfrac{\partial Q}{\partial x} = \sec^2 y$

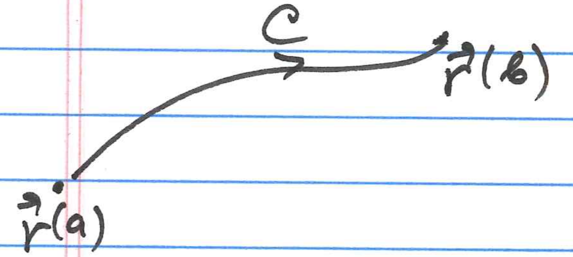
since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \Rightarrow \vec{F} = \langle \tan y, x\sec^2 y \rangle$

is conservative

$\therefore \displaystyle\int_C \tan y\, dx + x\sec^2 y\, dy$ is independent of path

Recall

$$\int_C \nabla f \cdot d\vec{r} \underset{\substack{\text{Thm for} \\ \text{line } \int}}{\overset{\text{Fundam.}}{=}} f(\vec{r}(b)) - f(\vec{r}(a))$$



Need to find potential function f.

$$\vec{F} = P\vec{i} + Q\vec{j} = \nabla f = f_x\vec{i} + f_y\vec{j}$$

$$f_x = P, \quad f_y = Q$$

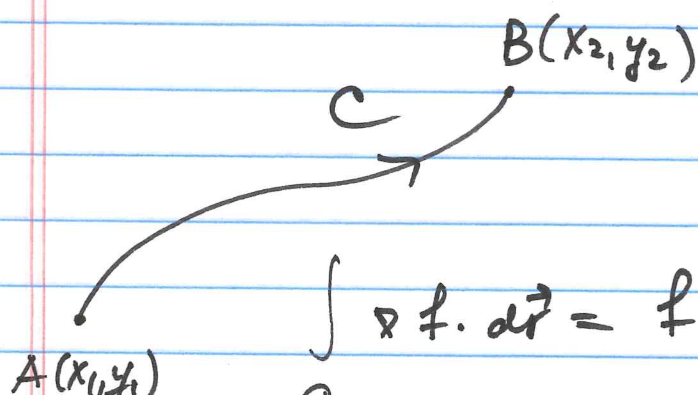
$$f_x = \tan y \ \underset{\text{wrt } x}{\overset{\text{integrate}}{\Rightarrow}} \ f(x,y) = x\tan y + \underbrace{g(y)}_{\text{some function of } y}$$

$$f_y = x\sec^2 y + g'(y) = \underbrace{x\sec^2 y}_{Q}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = K = \text{const}$$

$$\therefore \boxed{f(x,y) = x\tan y + K}$$

$f(x,y)$



$$\int_C \nabla f \cdot d\vec{r} = f(x_2,y_2) - f(x_1,y_1)$$

Hence,

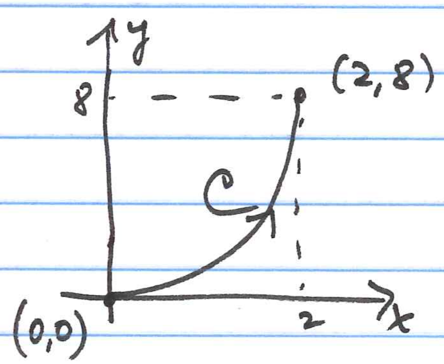
$$\int_C \tan y\,dx + x\sec^2 y\,dy = \left(x\tan y + K\right)\Big|_{x=2,\,y=\frac{\pi}{4}}$$

$$-\left(x\tan y + K\right)\Big|_{x=1,\ y=0} = 2\tan\frac{\pi}{4} - 1\cdot\cancel{\tan 0}^{\,0} =$$

$$= 2\cdot 1 - 1\cdot 0 = \boxed{2}$$

Ex Given force field $\vec{F}$, find work required to move object on the given oriented curve.

$\vec{F} = \langle y, x\rangle$, $\quad C$: parabola $y = 2x^2$ from $(0,0)$ to $(2,8)$



$$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt \qquad a \le t \le b$$

parabola $y = 2x^2$, $\quad 0 \le x \le 2$

$$\begin{cases} x = x \\ y = 2x^2 \end{cases} \qquad 0 \le x \le 2$$

or

$t = t$

$y = 2t^2$

$0 \le t \le 2$

$\vec{r}(x) = \langle x, 2x^2 \rangle$, $\vec{r}\,'(x) = \langle 1, 4x \rangle$

$$\therefore \; W = \int_C \langle y, x \rangle \cdot \vec{T}\, ds = \int_0^2 \underbrace{\langle 2x^2, x \rangle}_{\vec{F}} \cdot \underbrace{\langle 1, 4x \rangle}_{\vec{r}\,'} dx$$

$$= \int_0^2 (2x^2 + 4x^2)\, dx = \int_0^2 6x^2 dx = \frac{6}{3} x^3 \Big|_0^2 =$$

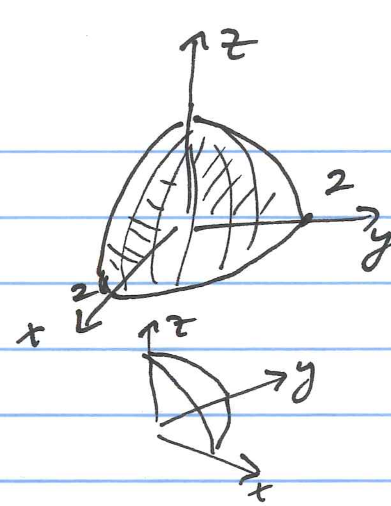
$$= 2 \cdot 8 = \boxed{16}$$

#19
15.7

$$\iiint_E (x+y+z)\,dV =$$

1^{st} octant: $x, y, z \geq 0$

paraboloid: $z = 4 - x^2 - y^2$



In cylindr. coordinates

$$z = 4 - x^2 - y^2 \quad \text{is} \quad z = 4 - r^2$$

Intersection of paraboloid w/ plane $z=0$:

$$\underset{0}{z} = 4 - x^2 - y^2 \ \ \& \ \ z = 0$$

$\Rightarrow 4 - x^2 - y^2 = 0$ or $x^2 + y^2 = 4$: circle centered at origin w/ rad $= 2$

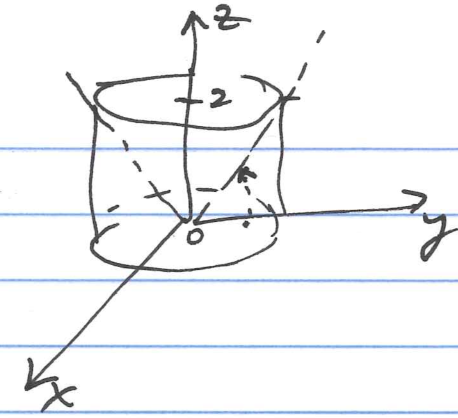
$$E = \{(r, \theta, z):\ 0 \leq r \leq 2,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq z \leq 4 - r^2\}$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \int_{z=0}^{4-r^2} \Big(\underbrace{r\cos\theta}_{x} + \underbrace{r\sin\theta}_{y} + z\Big)\, r\, dz\, dr\, d\theta$$

(2)

#21
15.7

$$\iiint_E x^2\,dV =$$



$x^2+y^2=1$: cylinder

$z=0$: plane

$z^2=4x^2+4y^2$: cone

Intersection: $x^2+y^2=1$ & $z^2=4x^2+4y^2$

$$z^2=4\underbrace{(x^2+y^2)}_{=1}$$

$$\Rightarrow z^2=4 \Rightarrow z=\pm 2$$

we take $z=2$

Cylindrical coordinates:

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z$$

$$x^2+y^2=r^2$$

Cylinder: $x^2+y^2=1 \Rightarrow r^2=1$ or $\boxed{r=1}$

Plane: $\boxed{z=0}$

Cone: $z^2=4x^2+4y^2$ or $z^2=4\underbrace{(x^2+y^2)}_{=r^2}$

$$\Rightarrow z^2=4r^2 \text{ or } \boxed{z=2r}$$

since $z\geq 0$

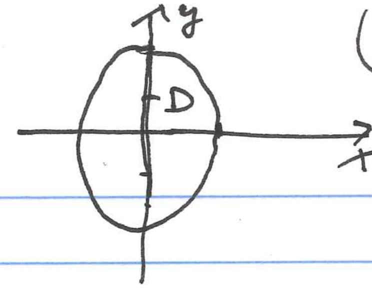
$$E=\{(r,\theta,z):\ 0\leq r\leq 1,\ 0\leq\theta\leq 2\pi,\ 0\leq z\leq 2r\}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{2r} \underbrace{r^2\cos^2\theta}_{x^2}\cdot r\,dz\,dr\,d\theta$$

3

#23
14.8

$f(x,y) = e^{-xy}, \quad x^2+4y^2 \le 1$



$f_x = e^{-xy}\cdot(-y)$

$f_x = f_y = 0 \Rightarrow x = y = 0$

$f_y = e^{-xy}\cdot(-x)$

$\Rightarrow \underset{\in D}{(0,0)}$ is a crit. pt

$f_{xx} = e^{-xy}\cdot y^2$

$f_{yy} = e^{-xy}\cdot x^2$

$f_{xy} = -\left[e^{-xy}(-x)\cdot y + e^{-xy}\cdot 1\right]$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(0,0)} = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

$\Rightarrow (0,0)$ is a saddle pt
i.e. no extremum at $(0,0)$

Boundary: $\underbrace{x^2+4y^2}_{g(x,y)}=1$

$f(x,y) = e^{-xy} \rightarrow$ max or min ?

$g_x = 2x, \quad g_y = 8y$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2+4y^2=1 \end{cases}$$

$$\begin{cases} -ye^{-xy} = \lambda \cdot 2x \\ -xe^{-xy} = \lambda \cdot 8y \\ x^2+4y^2=1 \end{cases} \quad \div$$

$$\frac{y}{x} = \frac{1}{4}\frac{x}{y} \Rightarrow y \cdot 4y = x^2$$

$$4y^2 = x^2$$

$$\Rightarrow x^2 + \underbrace{4y^2}_{= x^2} = 1 \Rightarrow 2x^2 = 1 \quad \text{or} \quad x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

$$\underbrace{x^2}_{= \frac{1}{2}} + 4y^2 = 1 \Rightarrow \frac{1}{2} + 4y^2 = 1$$

$$4y^2 = \frac{1}{2} \Rightarrow y^2 = \frac{1}{8}$$

$$y = \pm\frac{1}{2\sqrt{2}}$$

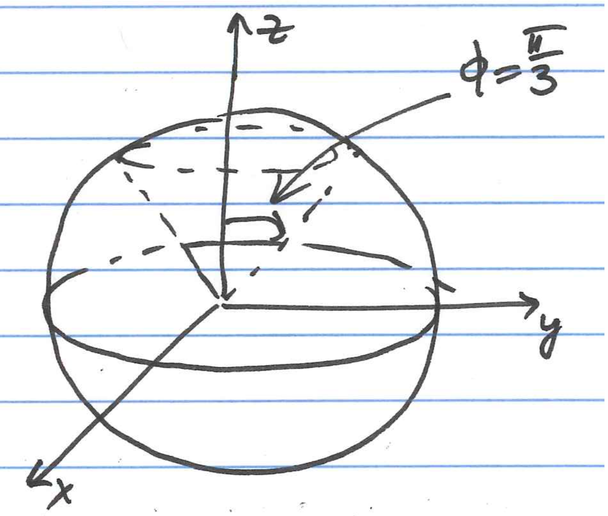
$\Rightarrow$ pts: $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$

Evaluate f at each of 4 points and pick f_{max} and f_{min}.

#22
S15.8

Evaluate $\iiint_E y^2 z^2\, dV$, where E lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 1$.



$E = \{(\rho, \theta, \phi):\ 0 \le \rho \le 1,$
$0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3}\}$

Hence

$$\iiint_E y^2 z^2\, dV =$$

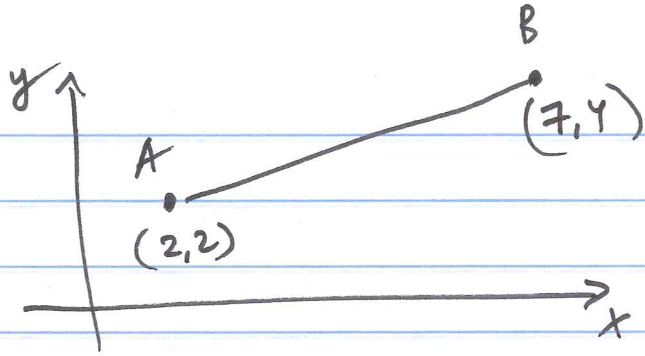
$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=0}^{1} (\rho \sin\phi \cos\theta)^2 (\rho\cos\phi)^2\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\, d\theta \cdot \int_{\phi=0}^{\pi/3} \sin^3\phi \cdot \cos^2\phi\, d\phi \cdot \int_{\rho=0}^{1} \rho^6\, d\rho$$

$$\int_{\theta=0}^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta = \ldots$$

$$\int_{\phi=0}^{\pi/3} \sin^3\phi \cdot \cos^2\phi\, d\phi = \int_{\phi=0}^{\pi/3} \sin\phi \cdot \underbrace{\sin^2\phi}_{1-\cos^2\phi} \cos^2\phi\, d\phi = \ldots$$

$u = \cos\phi$



$\vec{r}_0 = \langle 2,2 \rangle, \quad \vec{r}_1 = \langle 7,4 \rangle$

$$\vec{r}(t) = (1-t)\,\vec{r}_0 + t\,\vec{r}_1 = (1-t)\langle 2,2 \rangle + t\langle 7,4 \rangle$$

$$= \langle \underbrace{2(1-t)+7t}_{x(t)}, \underbrace{2(1-t)+4t}_{y(t)} \rangle$$

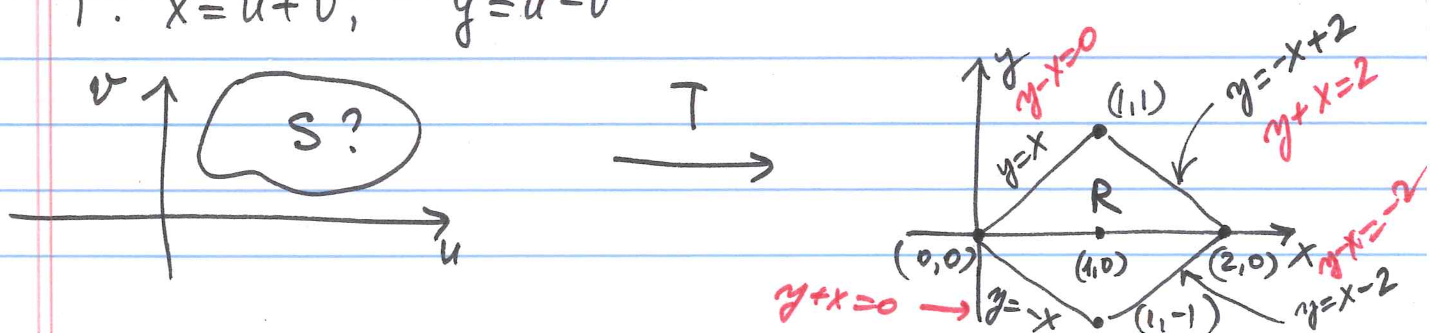
Segment $\vec{AB}$: $\quad x = 2(1-t)+7t = 5t+2$
$\qquad y = 2(1-t)+4t = 2t+2 \qquad 0 \le t \le 1$

Review example on work w/ variable force.

Ex Evaluate $\displaystyle\iint_R xy\,dA$

R: square w/ vertices $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$.

T: $x = u+v, \quad y = u-v$



Recall

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$|J|$ Jacobian of transformation T

$x = u+v \quad \Big| + \Big| -$

$y = u-v$

$$\begin{cases} u = \frac{1}{2}(x+y) \\ v = \frac{1}{2}(x-y) \end{cases} \Rightarrow$$

$y-x=0 \Rightarrow v=0$

$y+x=2 \Rightarrow u = \frac{1}{2}\cdot 2 = 1$

$y-x=-2 \Rightarrow v = \frac{1}{2}(2) = 1$

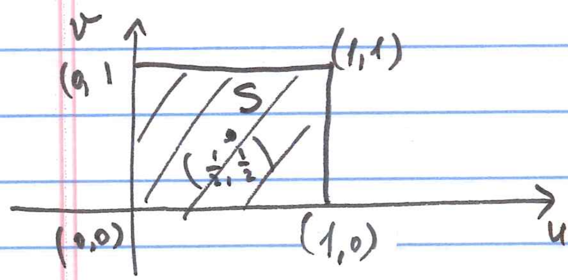
$x-y=2$

$y+x=0 \Rightarrow u=0$

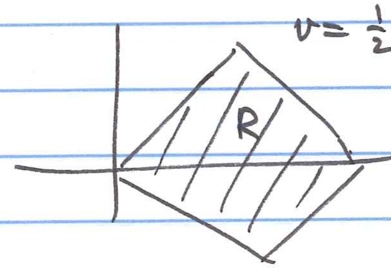
$\therefore \quad S = \{(u,v) : \ 0 \le u \le 1, \quad 0 \le v \le 1\}$

$x=1,\ y=0 \Rightarrow u = \frac{1}{2}(x+y) = \frac{1}{2}$

$v = \frac{1}{2}(x-y) = \frac{1}{2}$



$\xrightarrow{T}$



$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 < 0$$

$\Rightarrow |J| = 2$

$$\Rightarrow \iint_R xy\,dA = \int_{u=0}^{1} \int_{v=0}^{1} (u+v)(u-v)\cdot 2\ dv\,du =$$

$$= 2\int_{u=0}^{1}\int_{v=0}^{1}\left(u^2 - v^2\right)dv\,du = 2\int_{u=0}^{1}\left(u^2 v - \frac{v^3}{3}\right)\Big|_{v=0}^{1} du$$

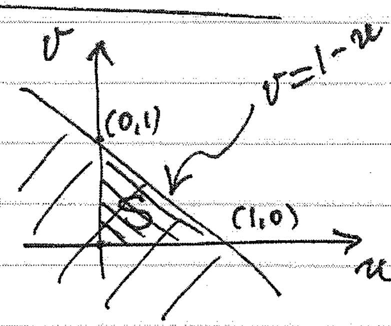
$$= 2\int_{u=0}^{1}\left(u^2 - \frac{1}{3}\right)du = 2\left(\frac{u^3}{3} - \frac{1}{3}u\right)\Big|_{0}^{1} = 2(0) = 0$$

Ex $S = \{(u,v):\ v \le 1-u,\ u \ge 0,\ v \ge 0\}$

Find image of S in xy-plane under transformation T:

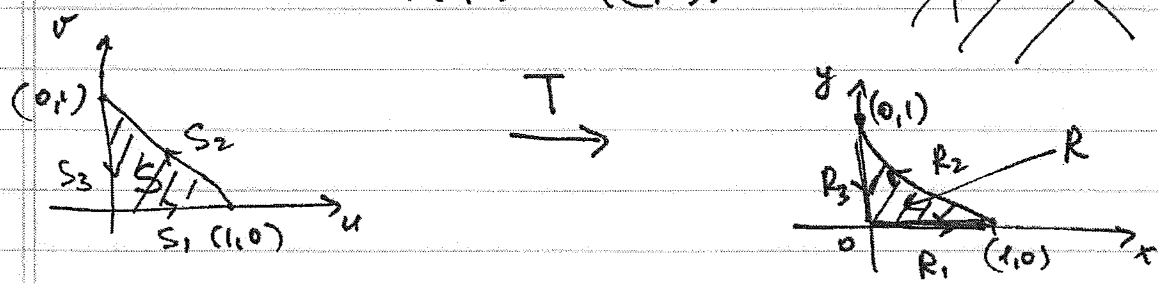
$T:\ x = u,\ y = v^2$

$S:\quad v \le 1-u$



$u=v=0 \Rightarrow \quad 0 \le 1-0$

$\Rightarrow (0,0) \in \{(u,v):\ v \le 1-u$



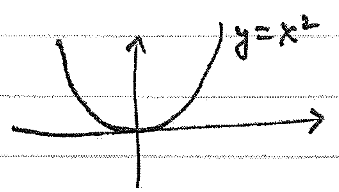
$S_1:\quad 0 \le u \le 1,\ v = 0 \quad \Rightarrow \quad x = u \Rightarrow 0 \le x \le 1$

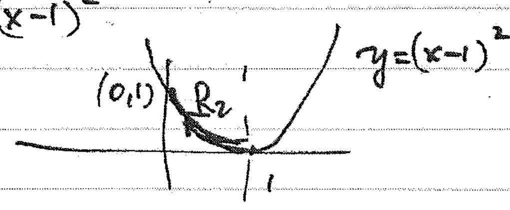
$u=0,\ v=0 \Rightarrow x=0,\ y=0 \qquad y = v^2 = 0^2 = 0$

$\Rightarrow \quad 0 \le x \le 1,\quad y = 0$

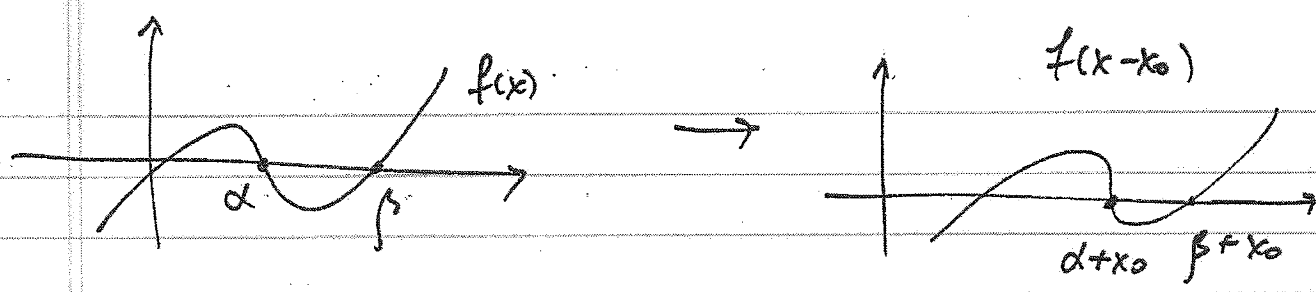
$S_2:\ v = 1-u \quad \Rightarrow \quad x = u,\ y = v^2 = (1-\underset{x}{u})^2 = (1-x)^2$

$y = (x-1)^2$





$f(x - x_0)$



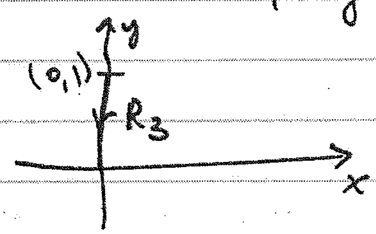
$y = (1-x)^2, \quad x = u, \quad y = v^2$

$v = 1-u \qquad u=1, v=0 \Rightarrow x=1, y=0 \Rightarrow (1,0)$

$u=0, v=1 \Rightarrow x=0, y=1 \Rightarrow (0,1)$

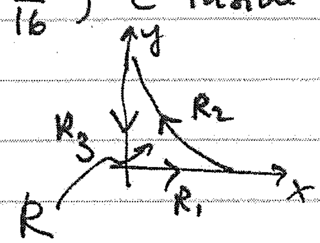
$S_3: \quad u=0, \; 0 \le v \le 1 \Rightarrow x=u=0, \; y=v^2$

$0 \le y \le 1$



$u=\frac{1}{4}, \; v=\frac{1}{4} \Rightarrow \left(\frac{1}{4}, \frac{1}{4}\right) \in S$

$x=u=\frac{1}{4}, \; y=v^2=\frac{1}{4^2}=\frac{1}{16} \Rightarrow \left(\frac{1}{4}, \frac{1}{16}\right) \in$ inside of



Ex Use a scalar line integral to find length of the curve

$\vec{r}(t) = \langle 20\sin\frac{t}{4}, 20\cos\frac{t}{4}, \frac{t}{2} \rangle, \quad 0 \le t \le 2$

$L = \int_0^2 |\vec{r}\,'(t)|\, dt \;=$

$\vec{r}\,' = \langle 20 \cdot \frac{1}{4}\cos\frac{t}{4}, -20\sin\frac{t}{4} \cdot \frac{1}{4}, \frac{1}{2} \rangle =$

$= \langle 5\cos\frac{t}{4}, -5\sin\frac{t}{4}, \frac{1}{2} \rangle$

$|\vec{r}\,'(t)| = \sqrt{\left(5\cos\frac{t}{4}\right)^2 + \left(-5\sin\frac{t}{4}\right)^2 + \left(\frac{1}{2}\right)^2} =$

$= \sqrt{25 + \frac{1}{4}} =$ since $\cos^2\frac{t}{4} + \sin^2\frac{t}{4} = 1$

$= \sqrt{\frac{101}{4}} = \frac{\sqrt{101}}{2}.$

$= \int_0^2 \frac{\sqrt{101}}{2} \cdot dt = \sqrt{101}$