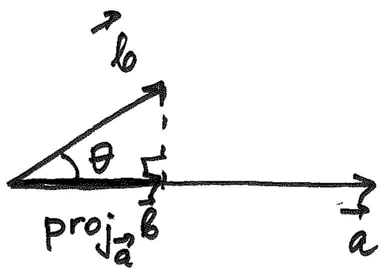
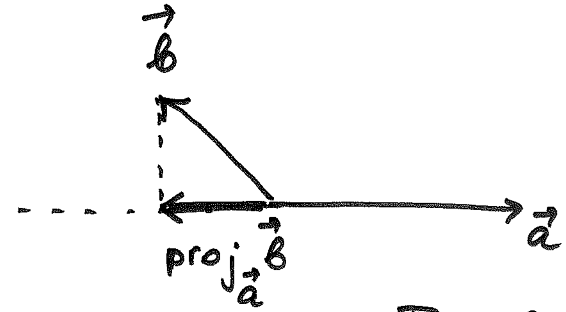
Projections (Cont'd)



$0 < \theta < \frac{\pi}{2}$

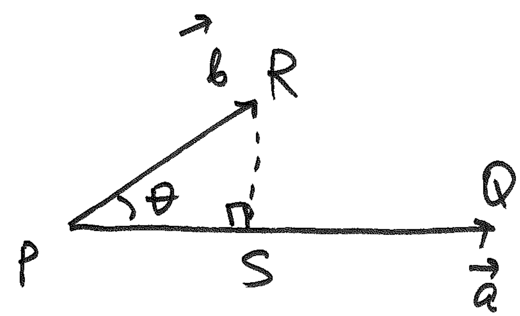


$\frac{\pi}{2} < \theta < \pi$

Component of $\vec{b}$ along $\vec{a}$ (scalar component of $\vec{b}$ along $\vec{a}$) is the signed magnitude of vector projection $\text{proj}_{\vec{a}}\vec{b}$.

Notation: $\text{comp}_{\vec{a}}\vec{b} = \pm |\vec{PS}|$

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = |\vec{b}| \cos\theta}$$



$|\vec{PS}| = |\vec{PR}| \cdot \cos\theta \qquad 0 < \theta < \frac{\pi}{2}$

$|\vec{PS}| = |\vec{b}| \cdot \cos\theta$

if $0 < \theta < \frac{\pi}{2} \Rightarrow \cos\theta > 0 \Rightarrow \text{comp}_{\vec{a}}\vec{b} = |\vec{PS}|$

if $\frac{\pi}{2} < \theta < \pi \Rightarrow \cos\theta < 0 \Rightarrow \text{comp}_{\vec{a}}\vec{b} = -|\vec{PS}|$

Recall

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot\underbrace{|\vec{b}|\cos\theta}_{\text{comp}_{\vec{a}}\vec{b}}$$

$$\Rightarrow \boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$: scalar projection of $\vec{b}$ along $\vec{a}$

Then

$$\text{proj}_{\vec{a}}\vec{b} = \text{comp}_{\vec{a}}\vec{b}\cdot\underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector in direction of } \vec{a}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|} = \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}}_{\text{scalar}}\cdot\underbrace{\vec{a}}_{\text{vector}}$$

$$\therefore \boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}}$$ vector projection of $\vec{b}$ onto $\vec{a}$

Recall from linear algebra:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2x2 matrix $\quad |A| = \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$: determinant of A

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ 3x3 matrix

element a_{ij} : i^{th} row, j^{th} column

$$|A| = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \underbrace{(-1)^{1+1}}_{=1} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \underbrace{(-1)^{1+2}}_{=-1} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +$$

$$+ a_{13} (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Note: any other row or column may be used to compute det A.

Ex

$$\begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 4 & 2 & -3 \end{vmatrix} = (-2) \underbrace{(-1)^{1+3}}_{=1} \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} + 0 \cdot \underbrace{(-1)^{2+3}}_{=-1} \begin{vmatrix} 1 & 5 \\ 4 & 2 \end{vmatrix} + (-3) \underbrace{(-1)^{3+3}}_{=1} \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} =$$

$$= -2\left(6 - (-4)\right) + 0 - 3(-1 - 15) = -2 \cdot 10 - 3(-16) = 28$$

12.4 CROSS PRODUCT (Vector Product)

Def The cross product / vector product of two vectors

$\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

is the vector defined

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\imath}\underbrace{(-1)^{1+1}}_{=1}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\jmath}\underbrace{(-1)^{1+2}}_{-1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} +$$

$$+ \hat{k}\underbrace{(-1)^{1+3}}_{1}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k} =$$

$$= (a_2b_3 - a_3b_2)\hat{\imath} - (a_1b_3 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$

<u>Ex</u> Find $\vec{a}\times\vec{b}$ and verify that $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$\vec{a}=\langle 1,-1,1\rangle, \qquad \vec{b}=\langle 1,1,1\rangle$

$$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - \hat{\jmath}\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{0} + \hat{k}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} =$$

$$= \hat{\imath}(-1-1) - \hat{\jmath}\underbrace{(1-1)}_{0} + \hat{k}(1-(-1)) = -2\hat{\imath}+2\hat{k} = \langle -2, 0, 2\rangle$$

To verify that $\vec{a}\times\vec{b}\perp\vec{a}$, we need to show that

$(\vec{a}\times\vec{b})\cdot\vec{a}=0$

dot product

$(\vec{a}\times\vec{b})\cdot\vec{a}=\langle -2,0,2\rangle\cdot\langle 1,-1,1\rangle = -2+0+2=0$ ✓

$\Rightarrow \vec{a}\times\vec{b}\perp\vec{a}$

Similarly for $\vec{b}$:

$$(\vec{a}\times\vec{b})\cdot\vec{b} = \langle -2, 0, 2\rangle \cdot \langle 1, 1, 1\rangle = 0 \Rightarrow \vec{a}\times\vec{b} \perp \vec{b}$$

Thm Cross product $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Note $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \vec{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \vec{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Consider

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} =$$

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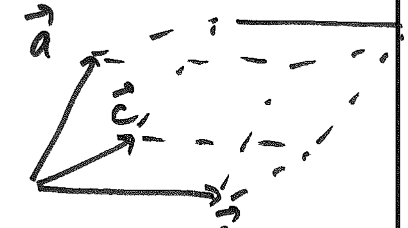
$= \vec{a}\cdot(\vec{b}\times\vec{c})$: <u>scalar triple product</u>



volume $= \vec{a}\cdot(\vec{b}\times\vec{c})$

<u>Direction</u> of $\vec{a}\times\vec{b}$ is determined by

right-hand rule: if fingers of the right hand curl from $\vec{a}$ to $\vec{b}$, thumb will point in the direction of $\vec{a}\times\vec{b}$.

<u>OR</u>

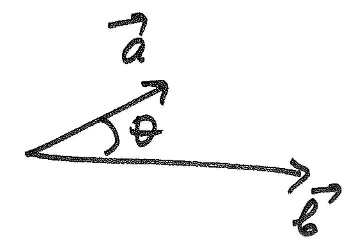
$\vec{a}$: along thumb

$\vec{b}$: along index finger

$\Rightarrow$ $\vec{a}\times\vec{b}$ will point along middle finger

<u>Thm</u> <u>Magnitude</u> / <u>length</u> of $\vec{a}\times\vec{b}$ is

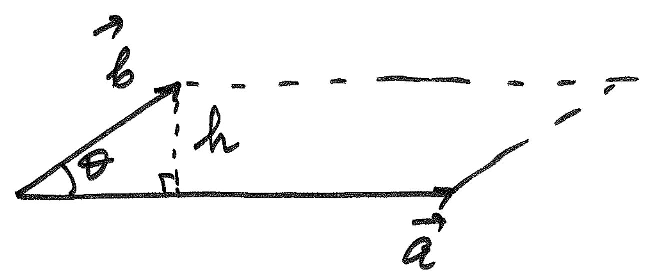
$$|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\theta$$



$0 \le \theta \le \pi$

Summary: cross product $\vec{a} \times \vec{b}$ of vectors $\vec{a}$ and $\vec{b}$ is the vector whose direction is determined by right hand rule and whose magnitude is $|\vec{a}| \cdot |\vec{b}| \sin\theta$.

Geometric interpretation of $|\vec{a} \times \vec{b}|$



Goal: find area of parallelogram determined by $\vec{a}$ and $\vec{b}$

$$A_{\square} = \underbrace{h}_{\text{height}} \cdot |\vec{a}|$$

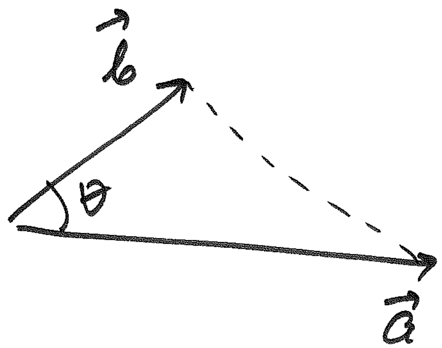
From right Δ: $h = |\vec{b}| \cdot \sin\theta$

$$\therefore \quad A_{\square} = h \cdot |\vec{a}| = |\vec{a}||\vec{b}| \sin\theta = |\vec{a} \times \vec{b}|$$

$$\therefore \quad \boxed{A_{\square} = |\vec{a} \times \vec{b}|}$$

$\Rightarrow$ area of parallelogram determined by $\vec{a}$ and $\vec{b}$ is the magnitude of $\vec{a} \times \vec{b}$.

Note



$$A_{\Delta} = \frac{1}{2} |\vec{a} \times \vec{b}| \; :$$ area of triangle determined by $\vec{a}$ and $\vec{b}$