Lecture 5

$$\hat{\imath}\times\hat{\jmath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\imath}\cdot 0 - \hat{\jmath}\cdot 0 + \hat{k}\cdot 1 = \hat{k}$$

$\hat{\imath}, \hat{\jmath}, \hat{k}$

Similarly $\boxed{\hat{\jmath}\times\hat{k} = \hat{\imath}}$ $\boxed{\hat{k}\times\hat{\imath} = \hat{\jmath}}$

Note: $$\hat{\jmath}\times\hat{\imath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \hat{\imath}\cdot 0 - \hat{\jmath}\cdot 0 + \hat{k}\cdot(-1) = -\hat{k}$$

$\Rightarrow \vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ but $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Note
$$\left.\begin{aligned} \hat{\imath}\times(\hat{\imath}\times\hat{\jmath}) &= \hat{\imath}\times\hat{k} = -\hat{\jmath} \\ (\hat{\imath}\times\hat{\imath})\times\hat{\jmath} &= \vec{0}\times\hat{\jmath} = \vec{0} \end{aligned}\right\} \Rightarrow (\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$$

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Properties of Cross Product

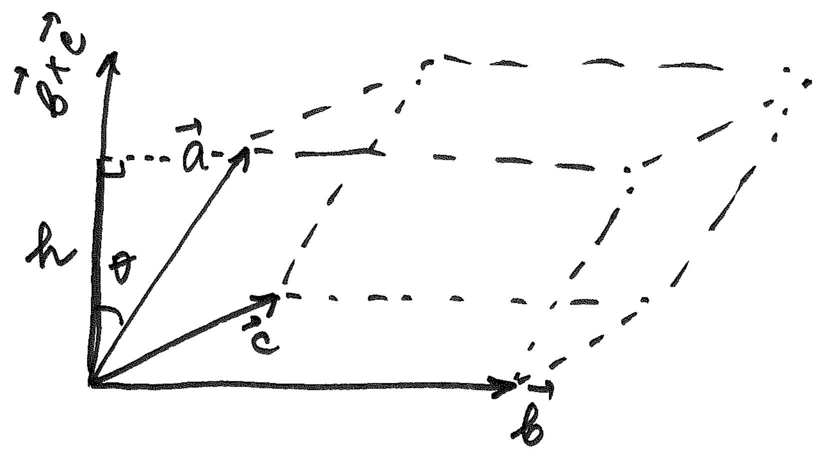
$\vec{a}, \vec{b}, \vec{c}$: vectors ; k: scalar

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$: triple scalar product
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Geometric interpretation of triple scalar product

Recall

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



<u>Goal:</u> find volume of parallelipiped

θ: angle between $\vec{a}$ and $\vec{b}\times\vec{c}$

$$\text{Volume} = \text{height} \cdot \text{area of base}$$

Area of base $= |\vec{b}\times\vec{c}|$

height $h = |\vec{a}|\cdot|\cos\theta|$

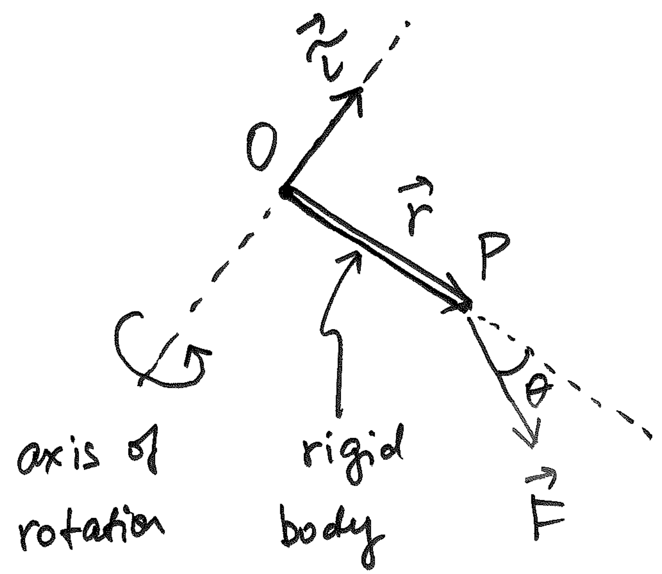
$$\therefore \text{Volume} = |\vec{a}|\cdot|\cos\theta|\cdot|\vec{b}\times\vec{c}| = |\vec{a}|\cdot|\vec{b}\times\vec{c}|\cdot|\cos\theta| = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$$

$$\Rightarrow \boxed{\text{Volume} = |\vec{a}\cdot(\vec{b}\times\vec{c})|}$$

<u>Note</u>: if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are in the same plane

$\Rightarrow$ they are called <u>coplanar vectors</u>

Torque



$\vec{r} = \vec{OP}$: position vector

Force $\vec{F}$ acts on a rigid body at some pt P given by the position vector $\vec{OP}$

Torque $\vec{\tau}$ (relative to origin) is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Torque measures tendency of a rigid body to rotate about the origin.

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| \cdot |\vec{F}| \cdot \sin\theta$$

magnitude of torque

θ: angle between $\vec{r}$ and $\vec{F}$

$0 \le \theta \le \pi$

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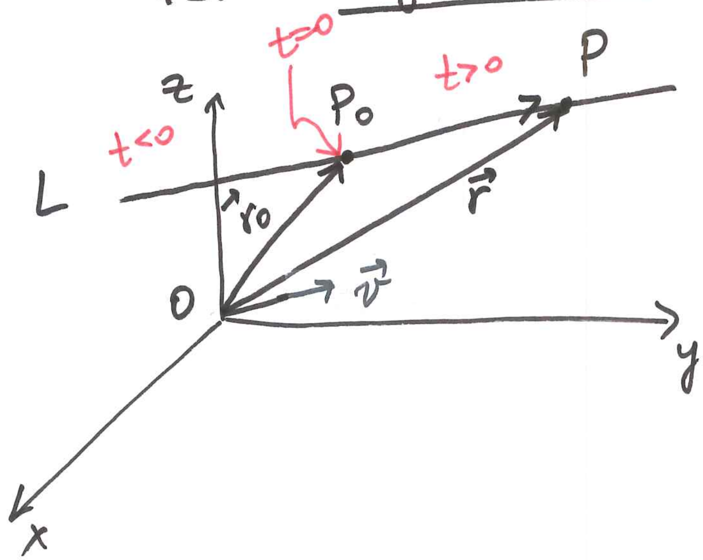
12.5 Equations of Lines and Planes



a line is determined by a pt $P_0(x_0, y_0, z_0)$ and direction given by vector $\vec{v}$.

L: line $\quad L \parallel \vec{v}$

$P(x, y, z)$

let $\vec{r_0} = \overrightarrow{OP_0}$, $\quad \vec{r} = \overrightarrow{OP}$

From Δ law: $\underbrace{\overrightarrow{OP_0}}_{=\vec{r_0}} + \overrightarrow{P_0P} = \underbrace{\overrightarrow{OP}}_{=\vec{r}} \Rightarrow \overrightarrow{P_0P} = \vec{r} - \vec{r_0}$

$\overrightarrow{P_0P} \parallel \vec{v} \Rightarrow \overrightarrow{P_0P} = \underbrace{t}_{\text{scalar}} \cdot \vec{v}$

$\Rightarrow \vec{r} = \vec{r_0} + \overrightarrow{P_0P} = \vec{r_0} + t\vec{v} \Rightarrow \boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ vector eq^n of the line through pt P_0 in direction of vector $\vec{v}$

$-\infty < t < \infty$

let $\vec{v} = \langle a, b, c \rangle$

$P_0(x_0, y_0, z_0) \Rightarrow \vec{OP_0} = \vec{r_0} = \langle x_0, y_0, z_0 \rangle$

$P(x, y, z) \Rightarrow \vec{OP} = \vec{r} = \langle x, y, z \rangle$

line L: $\vec{r} = \vec{r_0} + t\vec{v}$

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$$\therefore \boxed{x = x_0 + ta, \quad y = y_0 + t \cdot b, \quad z = z_0 + t \cdot c, \quad t \in \mathbb{R}}$$

<u>parametric equation</u> of a line through pt $P_0(x_0, y_0, z_0)$ in direction of $\vec{v} = \langle a, b, c \rangle$

a, b, c: <u>direction numbers</u> of line L

<u>Note</u> line can be defined using a different from P_0 pt on L and any other vector $\parallel \vec{v}$. Equation(s) will change but the line will be the same.

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e.g. $\vec{v}_1 = k\vec{v} = k\langle a, b, c\rangle = \langle ka, kb, kc\rangle$

some const

new direction numbers

$\vec{v}_1 \parallel \vec{v}$

Line:

$x = x_0 + t\cdot a, \quad y = y_0 + t\cdot b, \quad z = z_0 + t\cdot c$

We can eliminate parameter t:

$x = x_0 + t\cdot a \Rightarrow t = \dfrac{x - x_0}{a}$

$y = y_0 + t\cdot b \Rightarrow t = \dfrac{y - y_0}{b}$ $\quad a, b, c \neq 0$

$z = z_0 + t\cdot c \Rightarrow t = \dfrac{z - z_0}{c}$

$\therefore \boxed{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$

symmetric equations of line through pt $P_0(x_0, y_0, z_0)$ in direction $\vec{v} = \langle a, b, c\rangle$

<u>Note</u>: if one of a, b, c is zero, we still can write symmetric equations

let $a = 0 \Rightarrow x = x_0 + \cancel{at}^{\,0} \Rightarrow \boxed{x = x_0}$

$b, c \neq 0$

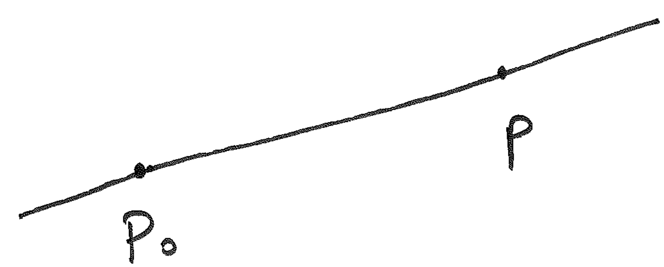
$\Rightarrow \boxed{\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$

$\therefore \boxed{x = x_0, \quad \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$

<u>Geometrically:</u> this line is in the plane parallel to yz-plane

<u>Ex</u> Find parametric and symmetric equations for the line through $(6, 1, -3)$ and $(2, 4, 5)$.

let $P_0(6, 1, -3)$, $P(2, 4, 5)$



$\overrightarrow{P_0P} \parallel \vec{v}$ and we can take $\vec{v} = \overrightarrow{P_0P}$

Note as pt P_0 we can choose either $(6,1,-3)$ or $(2,4,5)$

$P_0(6,1,-3) \Rightarrow x_0 = 6, \; y_0 = 1, \; z_0 = -3$

Parametric equations of the line

$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$

$\vec{v} = \langle a, b, c \rangle = \overrightarrow{P_0P} = \langle 2-6, 4-1, 5-(-3) \rangle = \langle -4, 3, 8 \rangle$

$\rightarrow a = -4, \quad b = 3, \quad c = 8$

$\therefore \boxed{x = 6 - 4 \cdot t, \quad y = 1 + 3t, \quad z = -3 + 8t, \quad t \in \mathbb{R}}$ — parametric eq^{ns} of the line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\Rightarrow \boxed{\frac{x-6}{-4} = \frac{y-1}{3} = \frac{z+3}{8}}$$ symmetric eq^{ns} of the line