Lecture 6

Note If a line goes through pts $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$, then its equation in the symmetric form is

$$\boxed{\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}}$$

Here $\vec{v} = \langle a, b, c\rangle = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$

Equation of a line segment

$\vec{v} = \overrightarrow{P_0P_1}$

We will use the previous example

$x = 6-4t,\quad y = 1+3t,\quad z = -3+8t$

$t=0:\quad x=6,\ y=1,\ z=-3 \Rightarrow P_0(6,1,-3)$

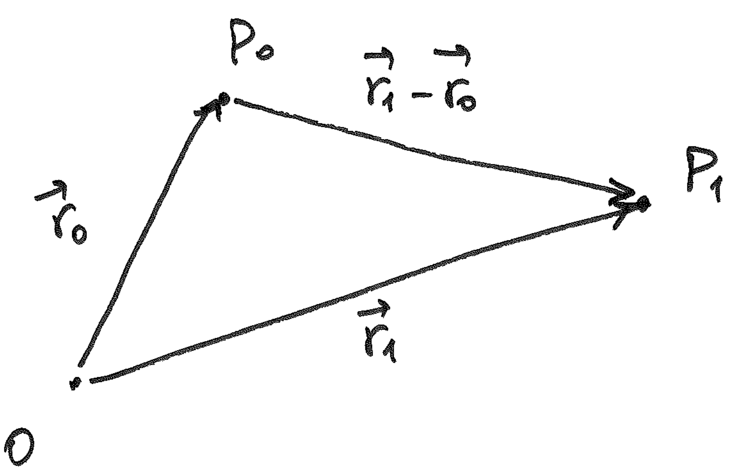
$t=1:\quad x=2,\ y=4,\ z=5 \Rightarrow P_1(2,4,5)$

$\Rightarrow$ segment P_0P_1 can be described by

$$\vec{r} = \vec{r_0} + t\underbrace{\vec{v}}_{=\overrightarrow{P_0P_1}} = \vec{r_0} + t\cdot\overrightarrow{P_0P_1}, \qquad 0 \le t \le 1 \qquad \vec{r_0} = \overrightarrow{OP_0}, \quad \vec{r} = \overrightarrow{OP}$$

or $\vec{r} = \langle x, y, z\rangle = \langle 6-4t,\ 1+3t,\ -3+8t\rangle \qquad 0 \le t \le 1$

OR



To define segment P_0P_1 we will use

$$\vec{r} = \vec{r_0} + t\vec{v} \quad \text{with} \quad \vec{v} = \overrightarrow{P_0P_1} = \vec{r_1} - \vec{r_0} \qquad 0 \le t \le 1$$

$$\vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$$

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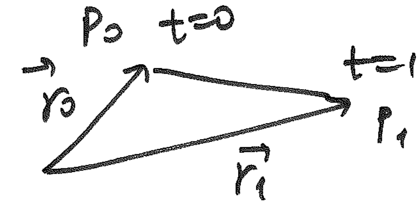
$$\boxed{\vec{r} = (1-t)\vec{r}_0 + t\,\vec{r}_1 \qquad 0 \le t \le 1}$$

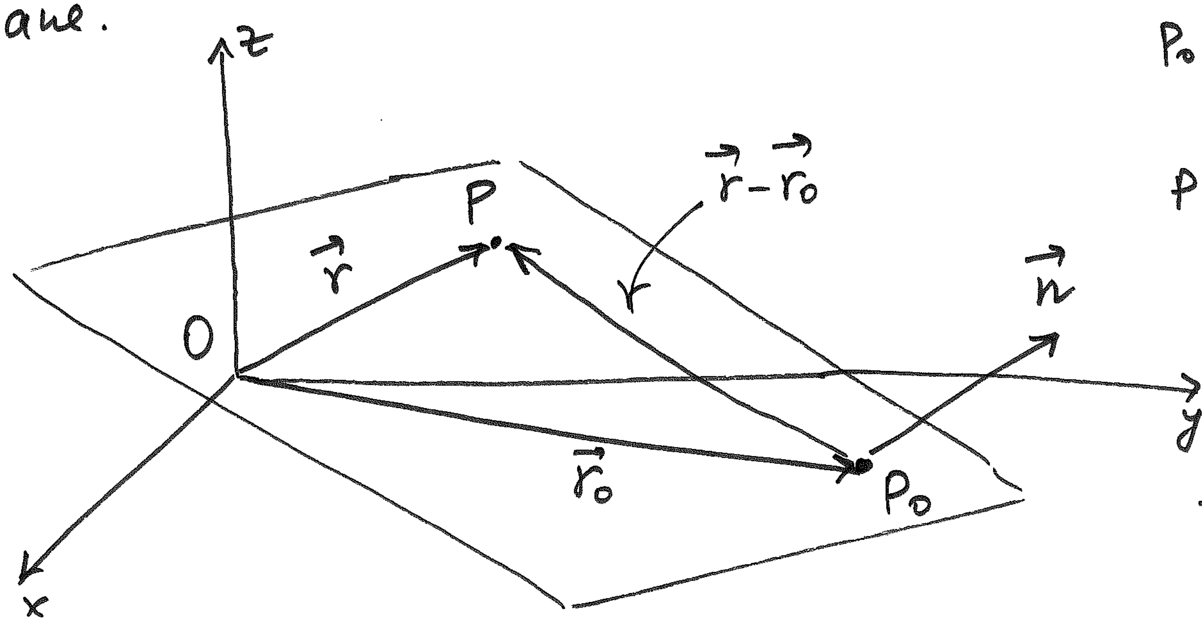
line segment P_0P_1



$t=0 \Rightarrow \vec{r} = \vec{r}_0$; $\quad t=1 \Rightarrow \vec{r} = \vec{r}_1$

PLANES

Plane is completely determined by a pt $P_0(x_0, y_0, z_0)$ in the plane and vector $\vec{n}$ that is normal (orthogonal) to the plane.



$P_0(x_0, y_0, z_0) \in$ plane
belongs

$P(x, y, z)$: arbitrary pt $\in$ plane

$\vec{n}$: normal vector

$\vec{r} = \overrightarrow{OP}$, $\quad \vec{r}_0 = \overrightarrow{OP_0}$

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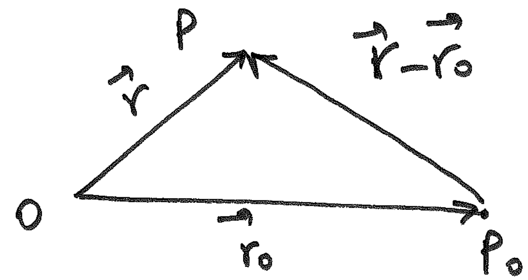
$P, P_0 \in$ plane $\Rightarrow \vec{P_0P} \in$ plane $\Rightarrow \vec{P_0P} \perp \vec{n} \Rightarrow \vec{P_0P} \cdot \vec{n} = 0$



or $\boxed{(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0}$

let $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$,

$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$

Then equation $(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$ can be written as

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

parametric equations of the plane through Pt $P_0(x_0, y_0, z_0)$ and normal $\vec{n} = \langle a, b, c \rangle$

We can write this equation as

$\boxed{ax + by + cz + d = 0}$

linear eq^{n} in x, y, z

where $d = -ax_0 - by_0 - cz_0$

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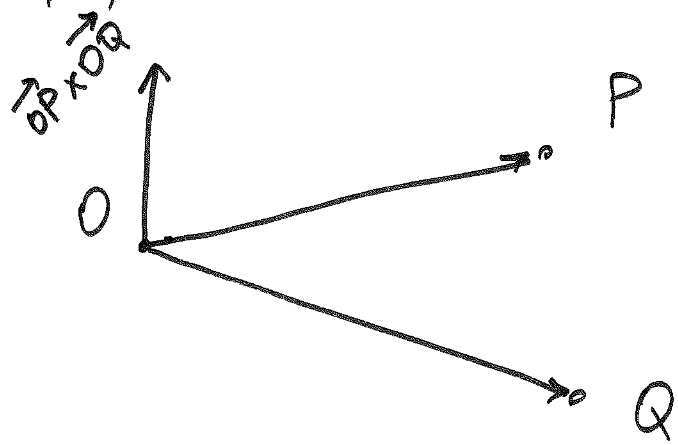
<u>Note</u> equation that is linear in x, y, z describes a plane.

<u>Ex</u> Find equation of the plane that goes through origin and points $(2,-4,6)$ and $(5,1,3)$.

$O(0,0,0)$ $\qquad$ $P(2,-4,6)$ $\qquad$ $Q(5,1,3)$



$O, P, Q \in$ plane

$\Rightarrow \vec{OP}, \vec{OQ} \in$ plane

$\Rightarrow \vec{OP} \times \vec{OQ} \perp$ plane

take $\vec{n} = \vec{OP} \times \vec{OQ}$

as a point $P_0 \in$ plane, we can any of O, P or Q

$$\vec{n} = \vec{OP} \times \vec{OQ} = \langle 2,-4,6\rangle \times \langle 5,1,3\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 6 \\ 5 & 1 & 3 \end{vmatrix} =$$

$$= -18\vec{i} + 24\vec{j} + 22\vec{k}$$

$\vec{n} = \langle a, b, c \rangle = \langle -18, 24, 22 \rangle$

For simplicity, let $P_0 = O(0,0,0)$.

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$-18(x-0) + 24(y-0) + 22(z-0) = 0$$

$$\boxed{-18x + 24y + 22z = 0}$$

Ex Find the point at which the line intersects the plane:

(1) $x = 1+2t, \quad y = 4t, \quad z = 2-3t$: line

(2) $x + 2y - z + 1 = 0$: plane

Point of intersection should satisfy eq^n of line and eq^n of the plane $\Rightarrow$ substitute (1) into (2):

$$(1+2t)+2\cdot(4t)-(2-3t)+1=0$$

$$13t=0 \Rightarrow t=0$$

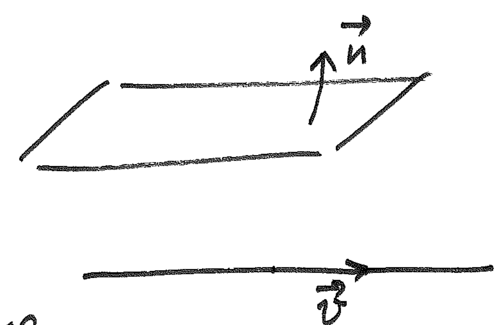
$\therefore \; x=1+2.0=1, \quad y=4.0=0, \quad z=2-3.0=2$

$\Rightarrow (1,0,2)$ is a pt of intersection.

<u>Note</u> line: $x=1+2t, \; y=4t, \; z=2-3t$ direction vector $\vec{v}=\langle 2,4,-3\rangle$

plane: $x+2y-z+1=0$ normal vector $\vec{n}=\langle 1,2,-1\rangle$

if line $\parallel$ plane $\Rightarrow$ we would need $\vec{n}\perp\vec{v}$, ie. $\vec{n}\cdot\vec{v}=0$



In our case,

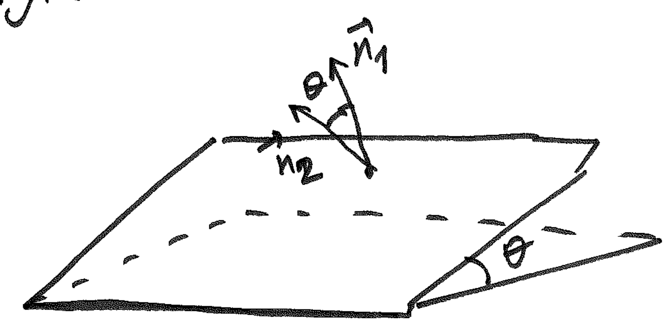
$\vec{n}\cdot\vec{v}=\langle 1,2,-1\rangle\cdot\langle 2,4,-3\rangle=13\neq 0$

$\Rightarrow \vec{n}\not\perp\vec{v} \Rightarrow$ line $\nparallel$ plane

Two planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$ are parallel if $\vec{n}_1 \parallel \vec{n}_2$, ie. $\vec{n}_1 = \underbrace{k}_{\text{const} \neq 0} \vec{n}_2$

If planes are not parallel, they intersect in a line.

Def Angle θ, $0 \le \theta \le \frac{\pi}{2}$, between two planes is the angle between their normal vectors.



Ex Determine whether planes are $\parallel$, $\perp$ or neither. If neither, find the angle between them.

$2x - 3y + 4z = 5$ $\qquad$ $x + 6y + 5z = 3$

$\vec{n}_1 = \langle 2, -3, 4 \rangle$ $\qquad$ $\vec{n}_2 = \langle 1, 6, 5 \rangle$

($ax + by + cz + d = 0$ plane w/ $\vec{n} = \langle a, b, c \rangle$)

$\vec{n_1} \nparallel \vec{n_2}$ since $\vec{n_1} \neq k\vec{n_2}$ (they are not proportional)

$\vec{n_1} \cdot \vec{n_2} = \langle 2, -3, 4\rangle \cdot \langle 1, 6, 5\rangle = 2 - 18 + 20 = 4 \neq 0 \Rightarrow \vec{n_1} \not\perp \vec{n_2}$

since $\vec{n_1} \nparallel \vec{n_2}$, $\vec{n_1} \not\perp \vec{n_2}$, planes intersect in a line at some angle θ that we can find by

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}| \cdot |\vec{n_2}|} = \frac{4}{\sqrt{2^2+(-3)^2+4^2} \cdot \sqrt{1^2+6^2+5^2}} = \frac{4}{\sqrt{29} \cdot \sqrt{62}}$$

$$\theta = \arccos\left(\frac{4}{\sqrt{29} \cdot \sqrt{62}}\right)$$