University of Idaho

Lecture 8

Quadric Surfaces (S 12.6 Cont'd)

Def A quadric surface is a surface whose equation is quadratic in x, y, z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

where $A, B, \dots, J$ are constants.

By rotation and translation, any quadric surface can be written as

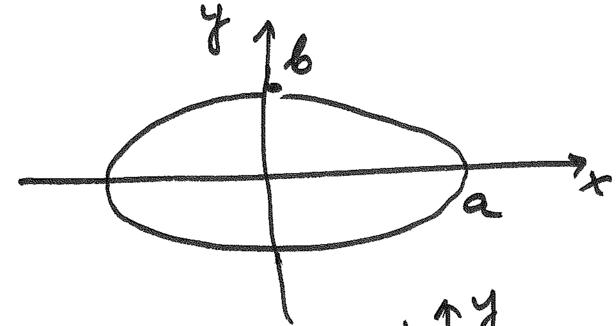
$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

Ex Use traces to sketch $9x^2 + 4y^2 + z^2 = 1$

Note: $\dfrac{x^2}{(\frac{1}{3})^2} + \dfrac{y^2}{(\frac{1}{2})^2} + \dfrac{z^2}{1} = 1$

Recall

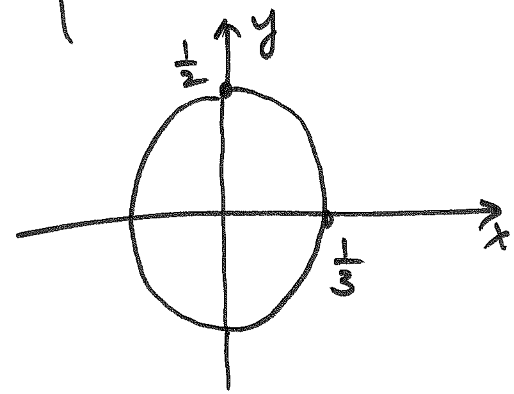
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: ellipse



Trace w/ $z=0$:

$9x^2+4y^2=1$ or $\frac{x^2}{(\frac{1}{3})^2} + \frac{y^2}{(\frac{1}{2})^2} = 1$

ellipse



Trace w/ $z=k$:

$\frac{x^2}{(\frac{1}{3})^2} + \frac{y^2}{(\frac{1}{2})^2} = 1 - k^2$

ellipse if $1-k^2>0$

$\Rightarrow k^2 < 1 \Rightarrow -1 < \underset{z}{\underset{\parallel}{k}} < 1$

Trace w/ $y=k$:

$\frac{x^2}{(\frac{1}{3})^2} + \frac{z^2}{1} = 1 - 4k^2$: ellipse if $1-4k^2>0$

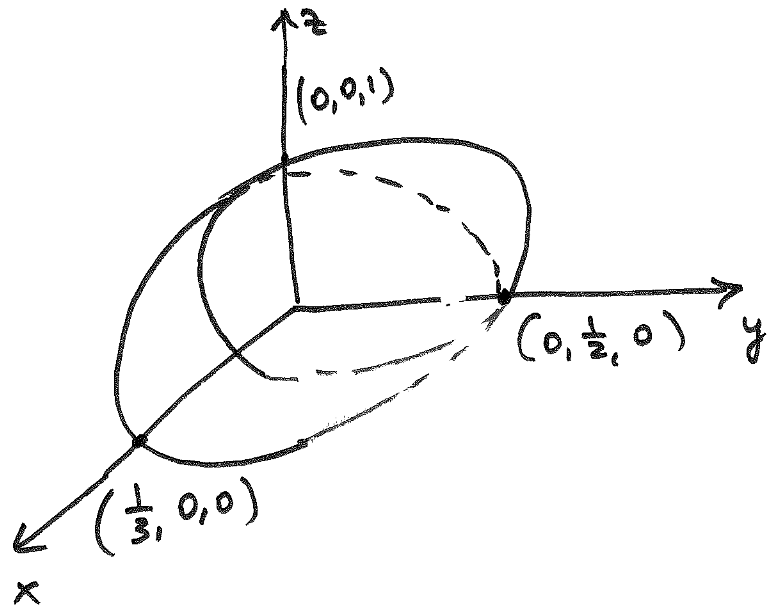
or $k^2 < \frac{1}{4} \Rightarrow -\frac{1}{2} < \underset{y}{\underset{\parallel}{k}} < \frac{1}{2}$

Trace w/ $x=k$:

$\frac{y^2}{(\frac{1}{2})^2} + \frac{z^2}{1} = 1 - 9k^2$: ellipse if $1-9k^2>0$

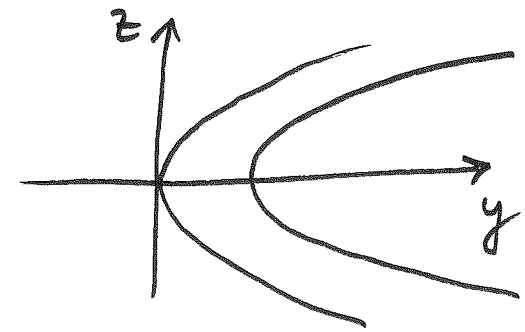
$\Rightarrow k^2 < \frac{1}{9} \Rightarrow -\frac{1}{3} < \underset{x}{\underset{\parallel}{k}} < \frac{1}{3}$

Surface: <u>ellipsoid</u> – all traces are ellipses

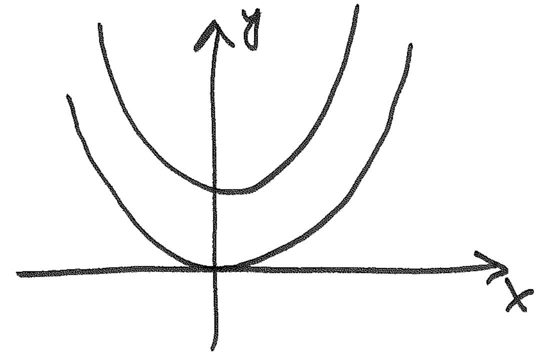


<u>Ex</u> Sketch surface $y = x^2 + 9z^2$

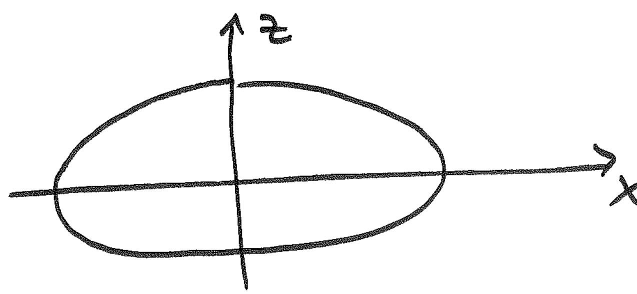
Trace w/ $x = k$: $y = 9z^2 + k^2$: parabolas

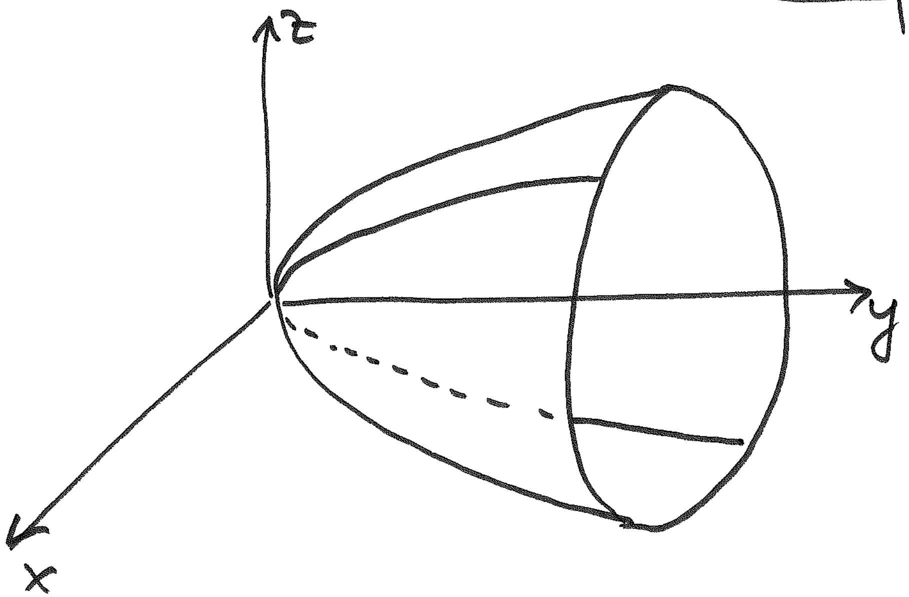


Trace w/ $z=k$: $y = x^2 + 9k^2$: parabolas



Trace w/ $y=k$: $x^2 + 9z^2 = k$: ellipse if $\underset{y}{\overset{\|}{k}} > 0$

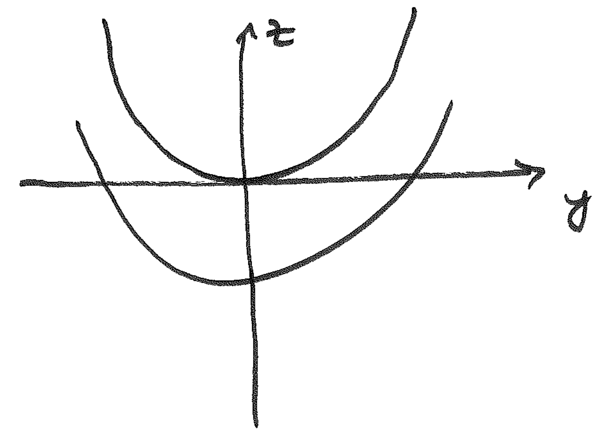




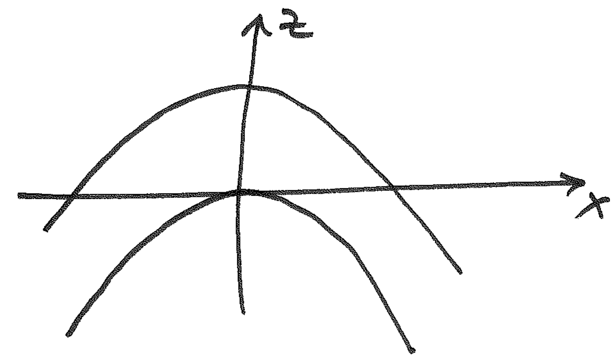
elliptic paraboloid

Ex $z = y^2 - x^2$

$x = k$: $z = y^2 - k^2$: parabolas



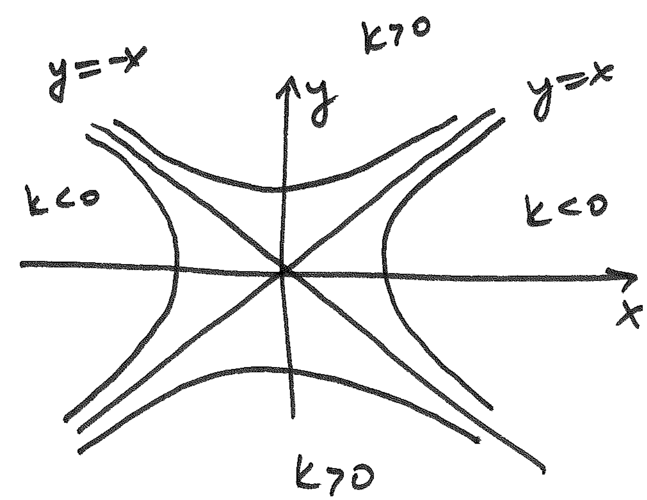
$y = k$: $z = -x^2 + k^2$: parabolas

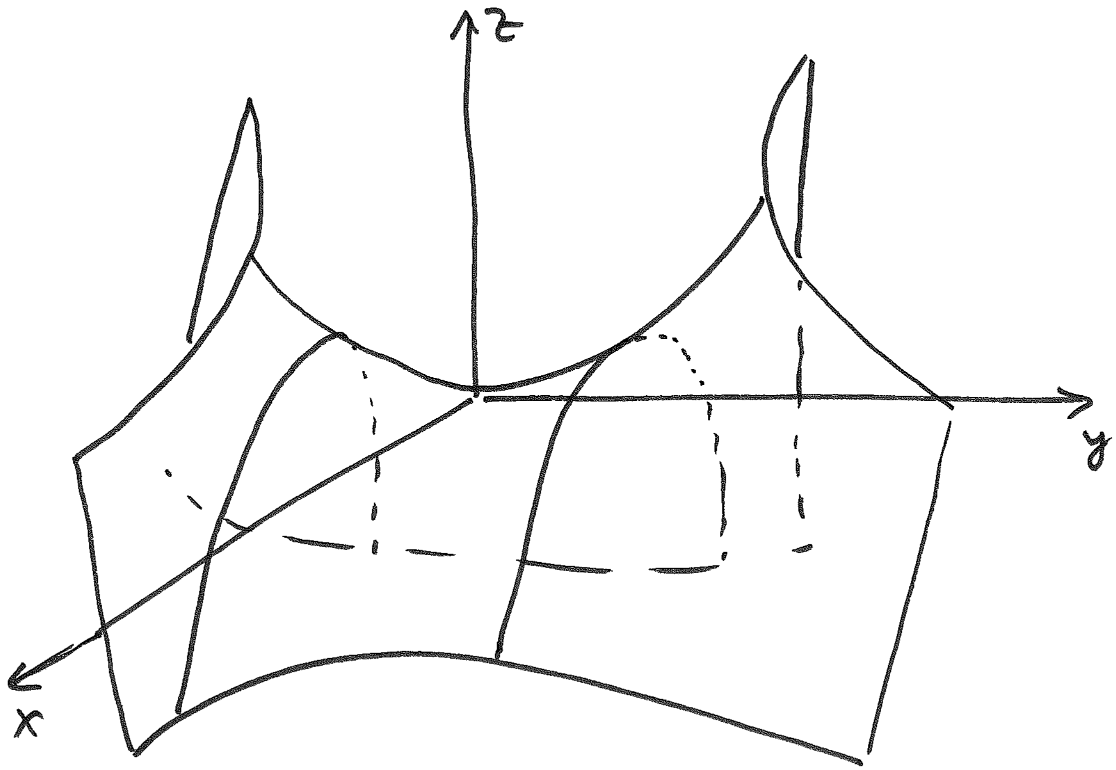


$z = k$: $y^2 - x^2 = k$

$k = 0$: $y^2 - x^2 = 0 \Rightarrow y = \pm x$: lines

$k \neq 0$: $y^2 - x^2 = k$: hyperbolas





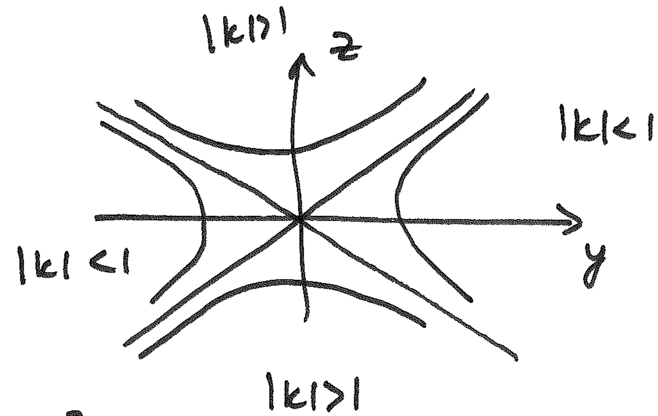
hyperbolic paraboloid

University of Idaho

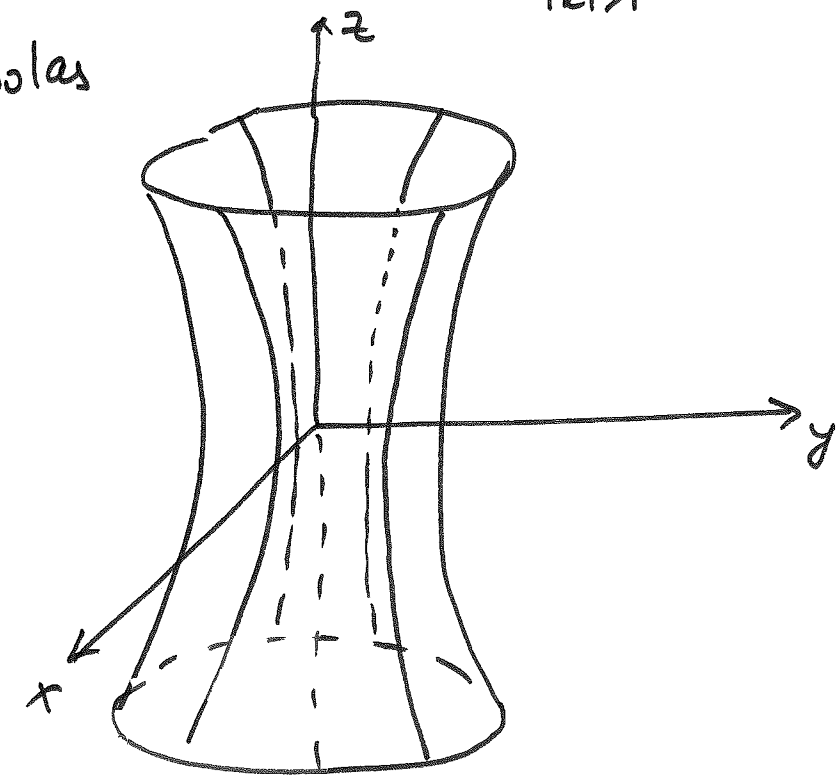
Ex $x^2 + \frac{y^2}{4} - z^2 = 1$

$z = k$: $x^2 + \frac{y^2}{4} = \underbrace{1 + k^2}_{>0}$: ellipse for all $k \; (=z)$ since $1 + k^2 > 0$

$x = k$: $\frac{y^2}{4} - z^2 = 1 - k^2$: hyperbolas



$y = k$: $x^2 - z^2 = 1 - \frac{k^2}{4}$: hyperbolas



hyperboloid of one sheet

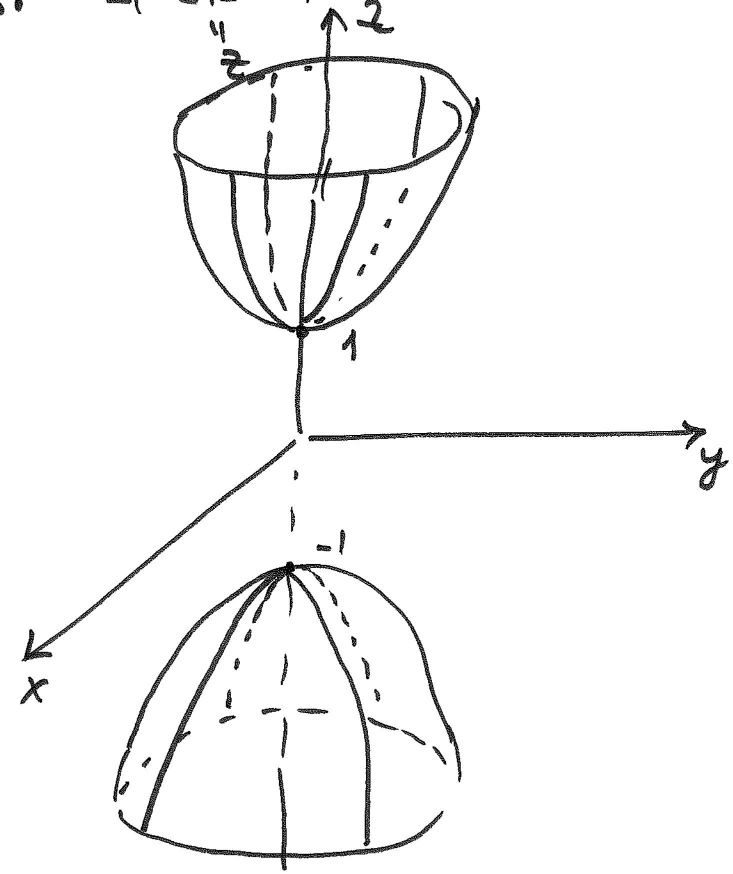
Ex $\quad -\frac{x^2}{4} - y^2 + z^2 = 1$

$z = k$: $\quad -\frac{x^2}{4} - y^2 = 1 - k^2 \quad$ or $\quad \frac{x^2}{4} + y^2 = k^2 - 1$: ellipse if

$k^2 - 1 > 0 \quad$ or $\quad |k| > 1$

$\Rightarrow \underset{k}{\overset{\shortparallel}{z}} > 1 \quad$ or $\quad \underset{k}{\overset{\shortparallel}{z}} < -1$

$\Rightarrow$ no points for $-1 < \underset{z}{\overset{\shortparallel}{k}} < 1$



$x = k$: $\quad -y^2 + z^2 = 1 + \frac{k^2}{4}$: hyperbolas

$y = k$: $\quad -\frac{x^2}{4} + z^2 = 1 + k^2$: hyperbolas

<u>hyperboloid of two sheets</u>

University of Idaho

Ex $\frac{z^2}{4} = x^2 + \frac{y^2}{9}$

$z = k$: $x^2 + \frac{y^2}{9} = \frac{k^2}{4}$: ellipse for $k > 0$

if $\underset{z}{\overset{\shortparallel}{k}} = 0 \Rightarrow x = y^2 = 0 \Rightarrow (0,0,0) \in$ surface

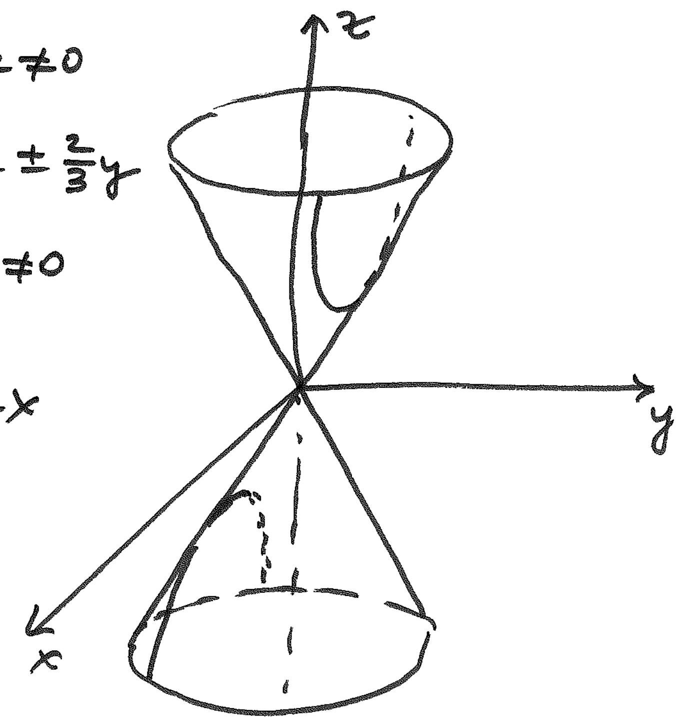
$x = k$: $\frac{z^2}{4} - \frac{y^2}{9} = k^2$: hyperbolas $k \neq 0$

lines $k = 0$ $z = \pm \frac{2}{3} y$

$y = k$: $\frac{z^2}{4} - x^2 = \frac{k^2}{9}$: hyperbolas $k \neq 0$

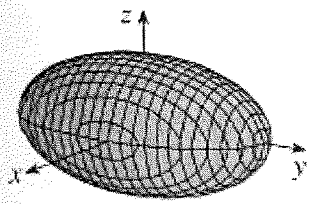
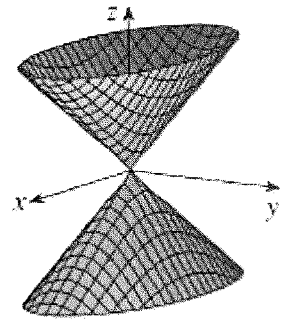
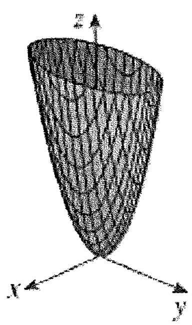
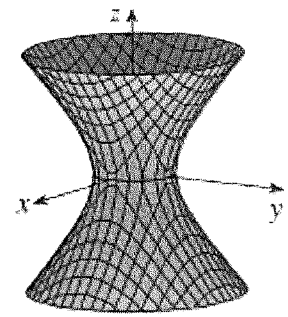
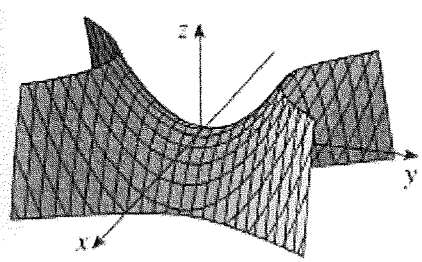
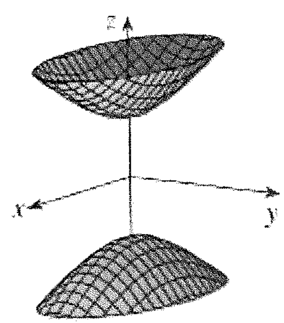
lines $k = 0$

$z = \pm 2x$



Cone

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

13.1 Vector Functions and Space Curves

Def A vector-valued function is a function whose domain is a set in $\mathbb{R}$ and whose range is a set of vectors.

$$\vec{r}(t) = \vec{r} = \langle f, g, h \rangle = \langle f(t), g(t), h(t) \rangle$$

or $\vec{r} : t \longrightarrow \langle f(t), g(t), h(t) \rangle$: vector-function

$$= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$