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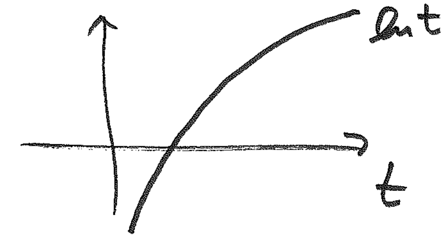
Lecture 9

13.1 Vector Functions and Space Curves

Ex $\vec{r}(t) = \langle \underbrace{t^4}_{f(t)}, \underbrace{e^{2t}\sin 5t}_{g(t)}, \underbrace{\ln t}_{h(t)} \rangle, \quad t > 0$



Def If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

if all limits exist.

Ex Find

$$\lim_{t \to 0} \left\langle \frac{e^t - 1}{t}, \frac{\sqrt{1+t} - 1}{t}, \frac{3}{1+t} \right\rangle =$$

$$= \left\langle \lim_{t \to 0} \frac{e^t - 1}{t}, \lim_{t \to 0} \frac{\sqrt{1+t} - 1}{t}, \lim_{t \to 0} \frac{3}{1+t} \right\rangle \boxed{=}$$

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$$\lim_{t\to 0}\frac{e^t-1}{t} = \frac{0}{0} \overset{\text{l'Hopital's}}{\underset{\text{rule}}{=}} \lim_{t\to 0}\frac{e^t}{1} = e^0 = \boxed{1}$$

$$\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t} \overset{0}{\underset{0}{=}} \lim_{t\to 0}\frac{(\sqrt{1+t}-1)(\sqrt{1+t}+1)}{t(\sqrt{1+t}+1)} = \lim_{t\to 0}\frac{(\cancel{1}+t)-\cancel{1}}{t(\sqrt{1+t}+1)} =$$

$$a^2-b^2=(a-b)(a+b)$$

$$= \lim_{t\to 0}\frac{\cancel{t}}{\cancel{t}(\sqrt{1+t}+1)} = \lim_{t\to 0}\frac{1}{\sqrt{1+t}+1} = \boxed{\frac{1}{2}}$$

$$\lim_{t\to 0}\frac{3}{1+t} = \boxed{3}$$

$$\boxed{=} \langle 1, \frac{1}{2}, 3\rangle$$

Def a vector function $\vec{r}(t)$ is <u>continuous</u> at $t=a$

if

$$\lim_{t\to a}\vec{r}(t) = \vec{r}(a)$$

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i.e. all components of $\vec{r}(t)$ are continuous at $t=a$.

<u>Def</u> a set of all points (x, y, z) given by

$x = f(t), \quad y = g(t), \quad z = h(t)$ <u>parametric eq$^{\text{ns}}$ of space curve C</u>

where $t \in I \subseteq \mathbb{R}$, is called a <u>space curve</u> C

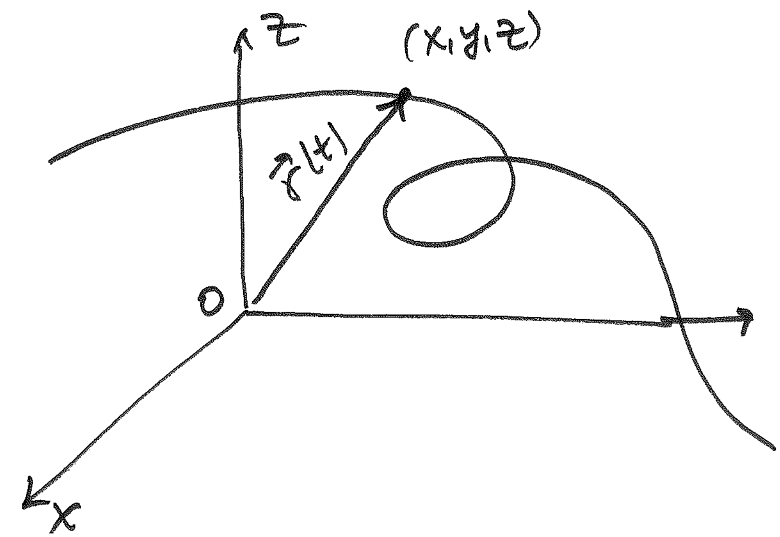
some interval

t: parameter

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$: vector eq$^{\text{n}}$ of space curve C

position vector

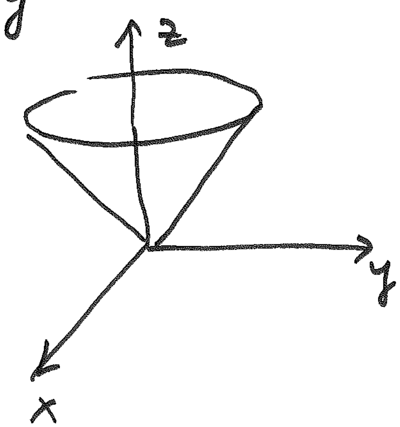
a tip of $\vec{r}(t)$ traces a space curve C as t varies.



Ex $\vec{r}(t) = \langle 4+t,\ 2-3t,\ -1+2t \rangle$

$x = 4+t,\ y = 2-3t,\ z = -1+2t$: line through pt $(4,2,-1)$ in direction of $\vec{v} = \langle 1,-3,2 \rangle$

Ex Find a vector function that represents the curve of intersection of the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 1+y$.



Cone: $z = \sqrt{x^2+y^2} \geq 0$

Plane: $z = 1+y$ or

$y - z + 1 = 0$: plane w/ $\vec{n} = \langle 0,1,-1 \rangle$

$ax+by+cz+d=0$ plane w/ $\vec{n} = \langle a,b,c \rangle$

To define a curve of intersection, we need to specify a parameter t.

Let $x = t$. **Q**: $y = g(t),\ z = h(t)$ - ?

$z = \sqrt{x^2+y^2}$ $\qquad$ $z = 1+y$

$z^2 = x^2+y^2 \quad \Rightarrow \quad (1+y)^2 = x^2+y^2$

$\parallel$

$(1+y)^2$

$1+2y+\cancel{y^2} = x^2+\cancel{y^2}$

$y = \frac{x^2-1}{2}$: parabola

Hence,

$x = t$

$y = \frac{x^2-1}{2} = \frac{t^2-1}{2}$

$z = 1+y = 1+\frac{t^2-1}{2} = \frac{t^2+1}{2}$

$\therefore \boxed{x = t, \quad y = \frac{t^2-1}{2}, \quad z = \frac{t^2+1}{2}, \quad t \in \mathbb{R}}$ <u>parametric</u> eqns of the curve of intersection

or $\boxed{\vec{r}(t) = \langle t, \frac{t^2-1}{2}, \frac{t^2+1}{2} \rangle, \quad t \in \mathbb{R}}$ vector eqn of the curve of intersection

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Note We can also use x as a parameter:

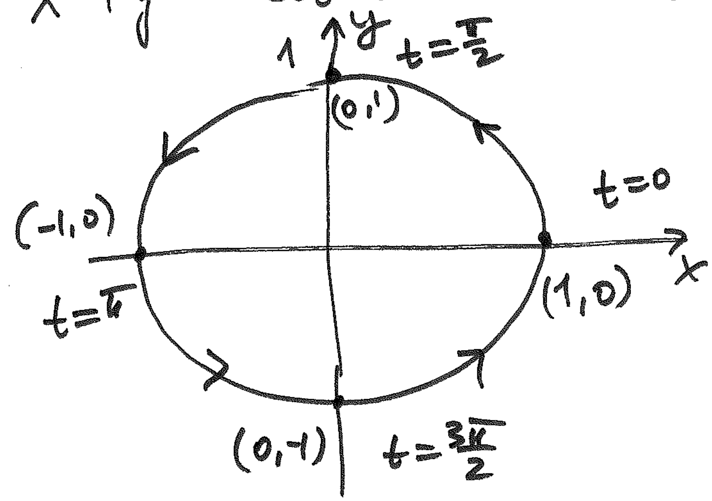
$\vec{r}(x) = \langle x, \frac{x^2-1}{2}, \frac{x^2+1}{2} \rangle$, $x \in \mathbb{R}$ x: parameter

Ex Sketch the curve

$\vec{r}(t) = \cos t \cdot \hat{\imath} + \sin t \cdot \hat{\jmath} + t \cdot \hat{k}$

i.e. $x = \cos t$, $y = \sin t$, $z = t$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow x^2+y^2 = 1$: circle centered at origin w/ rad $= 1$

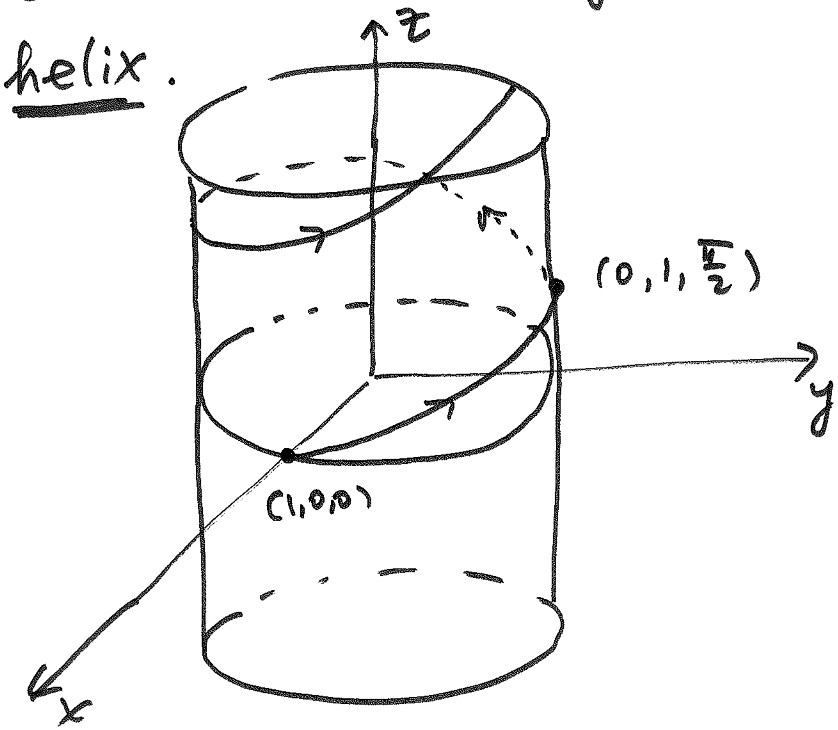


$t = 0 \Rightarrow x = \cos 0 = 1,\ y = \sin 0 = 0$
$\Rightarrow (1,0)$

$t = \frac{\pi}{2} \Rightarrow x = \cos\frac{\pi}{2} = 0,\ y = \sin\frac{\pi}{2} = 1$
$\Rightarrow (0,1)$

$\Rightarrow$ Curve lies on the vertical cylinder (since projections of this curve on xy-plane are the same circle).

as $t \nearrow$, $z \nearrow$ as well $\Rightarrow$ points move upward in positive z-direction but stay on the cylinder. This curve is called <u>**helix**</u>.



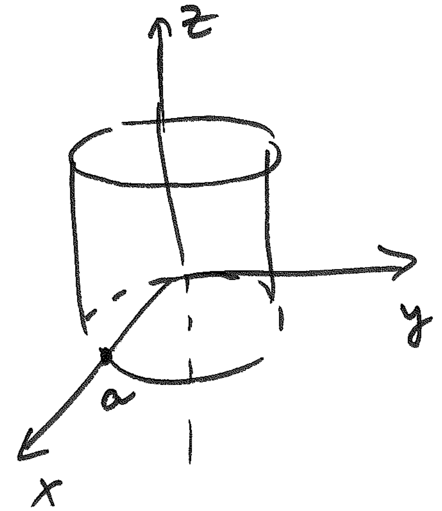
$t=0 \Rightarrow x=1,\ y=0,\ z=t=0 \Rightarrow (1,0,0)$

$t=\frac{\pi}{2} \Rightarrow x=0,\ y=1,\ t=\frac{\pi}{2}$

$(0,1,\frac{\pi}{2})$

<u>Note</u> as $t \nearrow$, curve spirals upward around the cylinder

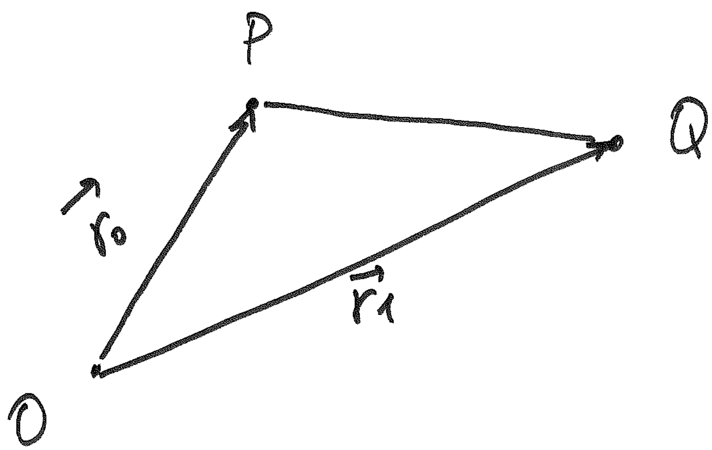
Note



$$x^2+y^2=a^2$$

$$x=a\cos t, \quad y=a\sin t$$

Ex Find a vector equation and parametric equations of the line segment that joins $P(1,0,1)$ and $Q(2,3,1)$.



$$\vec{r}_0=\vec{OP}=\langle 1,0,1\rangle$$

$$\vec{r}_1=\vec{OQ}=\langle 2,3,1\rangle$$

line segment PQ is

$$\vec{r}(t)=(1-t)\,\vec{r}_0+t\,\vec{r}_1, \qquad 0\le t\le 1$$

vector eq^n of the line segment

$\vec{r}(t) = (1-t)\langle 1,0,1\rangle + t\langle 2,3,1\rangle = \langle 1-t, 0, 1-t\rangle + \langle 2t, 3t, t\rangle$

$= \langle 1+t, 3t, 1\rangle$

$\Rightarrow \vec{r}(t) = \langle 1+t, 3t, 1\rangle \quad 0 \le t \le 1$ vector eq^t of segment PQ

Then

$x = 1+t, \quad y = 3t, \quad z = 1, \quad 0 \le t \le 1$ parametric eq^ns of line segment PQ

13.2 Derivatives and Integrals of Vector Functions

Def Derivative $\vec{r}\,'(t)$ of a vector function $\vec{r}(t)$ is

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

if this limit exists.