Lecture 10

Logistic Equation w/ Harvesting (Cont'd)

Ex $\frac{dx}{dt} = x(4-x) - h$: logistic equation w/ harvesting rate h

$x(t)$: population size (measured in hundreds)

$\frac{dx}{dt} = kx(M-x) - h$

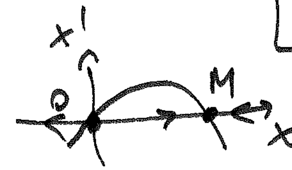
$k=1$, $M=4$

t: measured in years

If $h=0$, we have logistic equation : $\frac{dx}{dt} = x(4-x)$

then $\lim_{t\to\infty} x(t) = 4 = M$ (hundred) 400 fish

$$\boxed{\frac{dx}{dt} = kx(M-x)}$$



$x=0$, $x=M$

Now let $h=3$.

$$\frac{dx}{dt} = x(4-x) - 3$$

$$x(4-x) - 3 = 4x - x^2 - 3 = -(x^2 - 4x + 3) = -(x-1)(x-3)$$

$x_1 = 1$, $x_2 = 3$ $\qquad x_1 \cdot x_2 = 3$, $x_1 + x_2 = 4$

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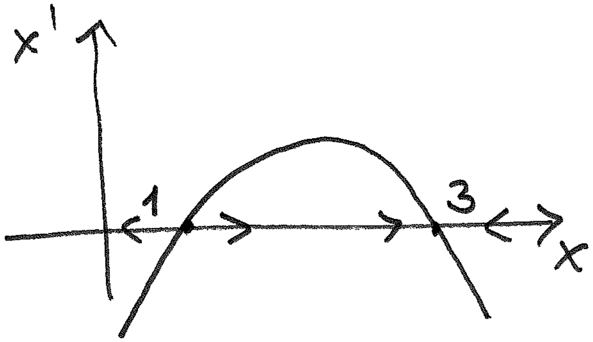
Two critical points: $x=1$ and $x=3$

$$\frac{dx}{dt} = -(x-1)(x-3)$$



phase diagram

$x=1$: unstable equil. sol$^{\underline{n}}$

$x=3$: stable equil. sol$^{\underline{n}}$

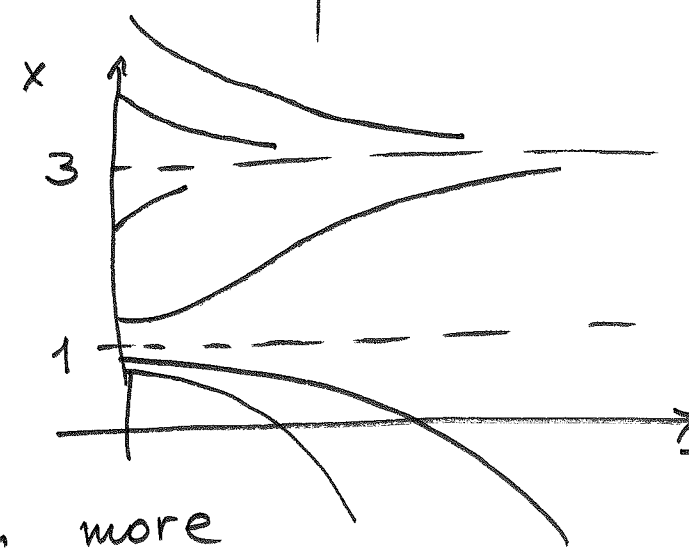
$ax^2+bx+c=0$

x_1, x_2: roots

$x_1 \cdot x_2 = \frac{c}{a}$

$x_1 + x_2 = -\frac{b}{a}$

Vietta's Thm



$x(t)=3$ is new limiting population size

$x(t)=1$ is threshold population

If lake is stocked initially with more 100 fish, then $x(t) \to 300$ fish as $t \to \infty$.

If lake is stocked initially w/ fewer than 100 fish, then lake will be "fished out" due to excessive harvesting within finite time.

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Consider again $\frac{dx}{dt} = \underbrace{x(4-x) - h}_{f(x)}$ — h harvesting rate

$x(4-x) - h = 0$

$-(x^2 - 4x + h) = 0$

Recall, critical points of $\frac{dx}{dt} = f(x)$ are solutions $x(t) = c$ for which $f(c) = 0$

$$x_{1,2} = \frac{4 \pm \sqrt{16 - 4h}}{2} = 2 \pm \sqrt{4-h}$$

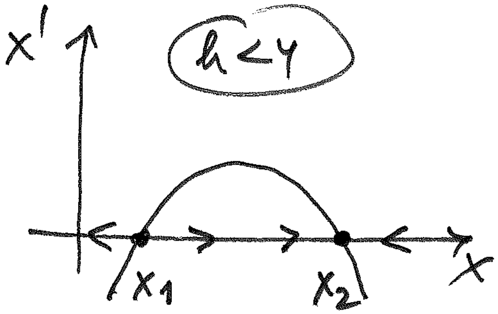
if $h < 4 \Rightarrow$ discriminant $4 - h > 0 \Rightarrow$ there are two real distinct roots

if $h = 4 \Rightarrow$ discriminant $4 - h = 0 \Rightarrow$ one real repeated root $x_{1,2} = 2$

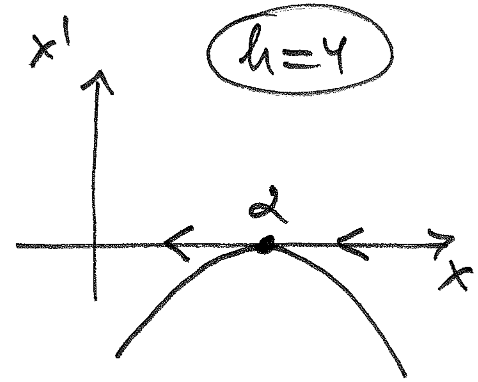
if $h > 4 \Rightarrow$ $4 - h < 0 \Rightarrow$ there no no real roots

phase diagram

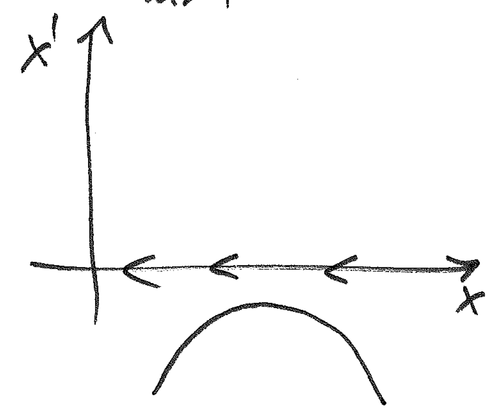
$h < 4$



$h = 4$



$h > 4$



$$\frac{dx}{dt} = x(4-x) - h$$

Crit. pts: $x(4-x) - h = 0$

$$-x^2 + 4x - h = 0$$

$$x^2 - 4x + h = 0 \qquad\qquad ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{4^2 - 4.1.h}}{2} = \qquad\qquad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 4h}}{2} =$$

$$= \frac{4 \pm 2\sqrt{4-h}}{2} = 2 \pm \sqrt{4-h}$$

(This is for $4-h > 0$ or $h < 4$)

If $h = 4 \Rightarrow x_{1,2} = 2 \pm 0 = 2$

If $h < 4 \Rightarrow$ roots are complex $\Rightarrow$ no real roots

We don't consider the case when $h=0$ on pg. 3.
We consider all possible cases as h varies.
If $h=0$, then

$$x_{1,2} = 2 \pm \sqrt{4} = 2 \pm 2 = \underset{x_1}{\underset{\parallel}{0}} \ \& \ \underset{x_2}{\underset{\parallel}{4}}$$

otherwise x_1, x_2 are different and they depend on h.
We have 3 cases: $h < 4$ (two distinct real roots), $h = 4$ (one root), $h > 4$ (no real roots)

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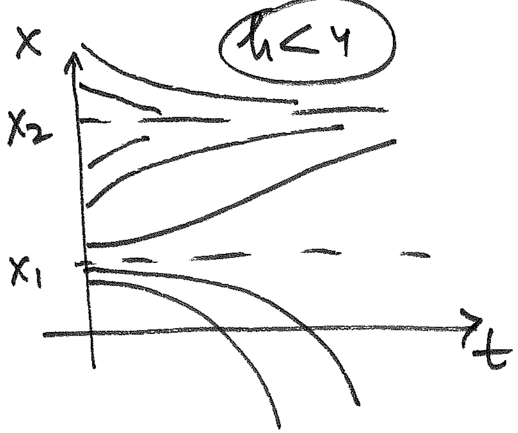
$h<4$

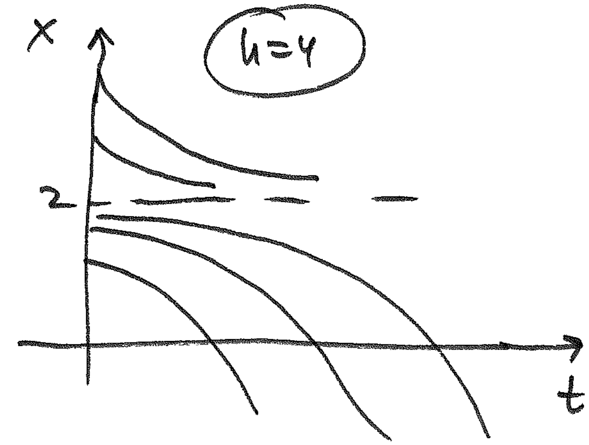


x_1 is unstable equil. sol^{n}

x_2 is stable —"—

2 equil. sol^{ns}

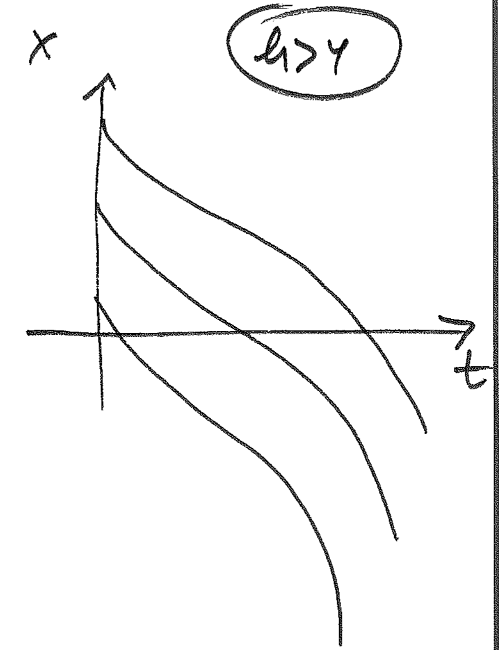
$h=4$



$x=2$ is unstable equil. sol^{n} (semi-stable)

1 equil. sol^{n}

$h>4$



no equil. solutions

Summary

$h<4$: two distinct equil. sol^{ns}

$h=4$: one equil. sol^{n}

$h>4$: no equil. sol^{ns}

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Def The value ($h=4$) for which qualitative behaviour of solutions of DE with parameter h changes as h increases is called a bifurcation point.

One of the ways to visualize this change in solution behaviour is to plot bifurcation diagram.

$$\frac{dx}{dt} = \underbrace{x(4-x)-h}_{f(x)}$$

$x(t)=c$: critical pt $\Rightarrow f(c)=0$

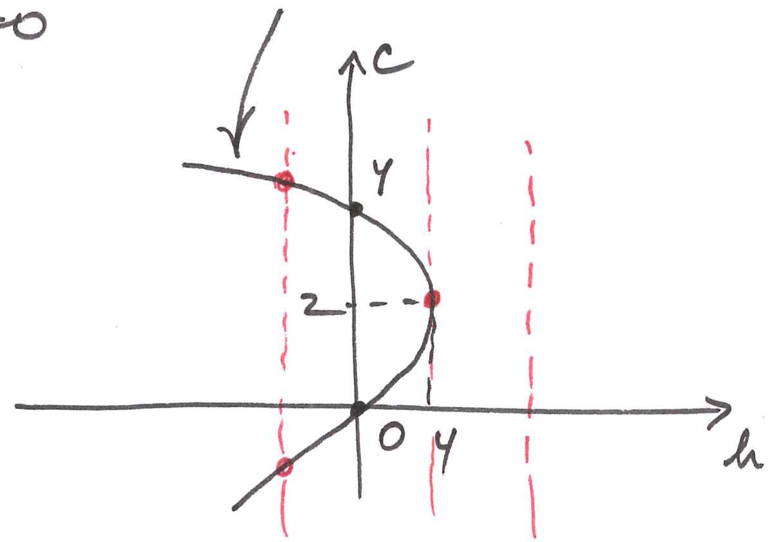
$c(4-c)-h=0$

or $\boxed{(c-2)^2 = 4-h}$

$h=0 \Rightarrow (c-2)^2 = 4$

$c = 2 \pm 2 = 4, 0$

$(c-2)^2 = 4-h$



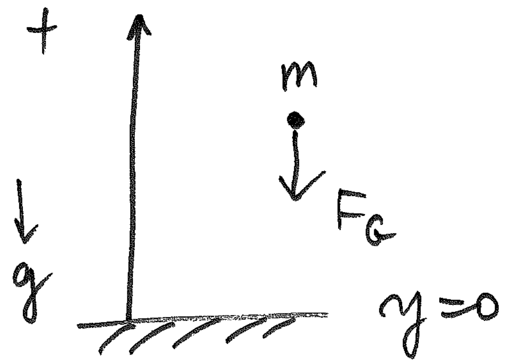
2.3 Acceleration-Velocity Models

Consider vertical motion under the action of gravity

$F = ma \qquad a = \frac{dv}{dt} = \frac{d^2y}{dt^2} \qquad F = F_G = -mg$ force due to gravity

$v(t)$: velocity

$y(t)$: vertical displacement



Now we will include air resistance.

$F = F_G + F_R$ force due to air resistance

$F_R = k v^p, \quad 1 \le p \le 2$

$k>0$: depends on size and shape of the body, density and viscosity of air

$p=1$: for small velocities

$p=2$: for high speeds

$1<p<2$: for intermediate speeds

Case 1 F_R is proportional to velocity

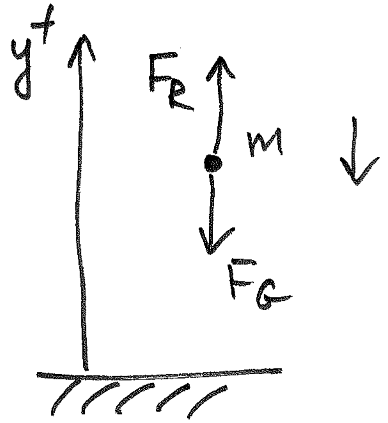
$F_R = -kv$

air resistance always acts in the direction opposite to direction of motion

Note

$v>0 \Rightarrow F_R<0$

$v<0 \Rightarrow F_R>0$



$$ma = F = F_G + F_R = -mg - kv$$

$$\frac{1}{m}\Big|\quad m\frac{dv}{dt} = mg - kv \quad : \quad 1^{st} \text{ order DE, linear}$$

$$\frac{dv}{dt} = \underbrace{-g}_{\text{acceleration due to gravity}} - \underbrace{\frac{k}{m}v}_{\text{acceleration due to air resistance}}$$

Let $\rho = \frac{k}{m}$: drag coefficient

$$\frac{dv}{dt} = -g - \rho v$$: separable DE

$$\frac{dv}{dt} = -\rho\left(v + \frac{g}{\rho}\right)$$