Lecture 11

Case 1. F_R is proportional to velocity (Cont'd)

$$\frac{dv}{dt} = -\rho\left(v + \frac{g}{\rho}\right) : \text{ separable } 1^{st} \text{ order DE}$$

$$\frac{dv}{v + \frac{g}{\rho}} = -\rho\, dt$$

$$\ln\left|v + \frac{g}{\rho}\right| = -\rho t + \tilde{C} \quad \Big| \exp$$

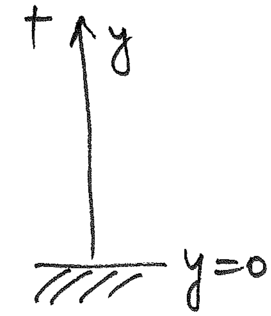
$$\left|v + \frac{g}{\rho}\right| = e^{-\rho t + \tilde{C}}$$

$$v + \frac{g}{\rho} = C e^{-\rho t}$$

IC: $v(0) = v_0 \Rightarrow$ at $t=0$: $\underbrace{v(0)}_{v_0} + \frac{g}{\rho} = C \underbrace{e^0}_{=1} \Rightarrow \boxed{C = v_0 + \frac{g}{\rho}}$

$$\boxed{v(t) = -\frac{g}{\rho} + \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t}} \quad , \quad \rho = \frac{k}{m} > 0$$

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\Big[-\frac{g}{\rho} + \big(v_0 + \frac{g}{\rho}\big)\underbrace{e^{-\rho t}}_{\to 0}\Big] = \underbrace{-\frac{g}{\rho}}_{= v_\tau} : \underline{\text{terminal velocity}}$$



$v_\tau = -\frac{g}{\rho}$: $\underline{\text{terminal velocity}}$

$|v_\tau| = \frac{g}{\rho}$: terminal speed $= \Big|\lim_{t\to\infty} v(t)\Big|$

$$\rho = \frac{k}{m} \Rightarrow \quad v_\tau = -\frac{g}{\frac{k}{m}} = -\frac{gm}{k}$$

$y(t)$: vertical displacement, $\quad \frac{dy}{dt} = v(t)$

$$y(t) = \int v(t)\,dt = \int \Big[\big(v_0 + \frac{g}{\rho}\big)e^{-\rho t} - \frac{g}{\rho}\Big]dt =$$

$$= -\frac{1}{\rho}\big(v_0 + \underbrace{\frac{g}{\rho}}_{=-v_\tau}\big)e^{-\rho t} - \frac{g}{\rho}t + C_2$$

IC: $y(0) = y_0$: initial displacement

$$\Rightarrow \underbrace{y(0)}_{= y_0} = -\frac{1}{\rho}\big(v_0 + \frac{g}{\rho}\big)\underbrace{e^0}_{=1} - \cancel{\frac{g}{\rho}\cdot 0}^{\,0} + C_2 \Rightarrow C_2 = y_0 + \frac{1}{\rho}\big(v_0 + \underbrace{\frac{g}{\rho}}_{=-v_\tau}\big)$$

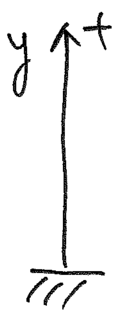
$$C_2 = y_0 + \frac{1}{\rho}(v_0 - v_\tau)$$

$$\therefore\ y(t) = -\frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + v_\tau \cdot t + y_0 + \frac{1}{\rho}(v_0 - v_\tau)$$

$$\boxed{y(t) = \frac{1}{\rho}(v_0 - v_\tau)\left[1 - e^{-\rho t}\right] + v_\tau \cdot t + y_0}$$

<u>Case 2</u> <u>F_R is proportional to v^2</u>

$F_R = -kv$



$F_R = \pm k v^2$

depends on sign of v

or $F_R = -k v \cdot |v|$

F_R always acts in the direction opposite to motion

$$m\frac{dv}{dt} = F_R + F_G$$

$$m\frac{dv}{dt} = -k v|v| - mg \quad \Big| \cdot \frac{1}{m}$$

$\rho = \frac{k}{m}$: <u>drag coefficient</u>

$$\frac{dv}{dt} = -\rho v|v| - g$$

Upward motion: $v > 0 \Rightarrow |v| = v$

$$|v| = \begin{cases} v, & v \geq 0 \\ -v, & v < 0 \end{cases}$$

$$\frac{dv}{dt} = -\rho v^2 - g$$

$$\frac{dv}{dt} = -\rho\left(v^2 + \frac{g}{\rho}\right) \quad : \text{ separable } 1^{st} \text{ order DE}$$

Recall

$$\int \frac{du}{u^2+1} = \arctan u + C$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\frac{dv}{v^2 + \underbrace{\frac{g}{\rho}}_{= a^2}} = -\rho \, dt$$

$$\frac{1}{\sqrt{\frac{g}{\rho}}}\arctan\frac{v}{\sqrt{\frac{g}{\rho}}} = -\rho t + C_1 \quad \Big| \cdot \sqrt{\frac{g}{\rho}}$$

$$\arctan\sqrt{\frac{\rho}{g}}\, v = -\sqrt{\rho g}\, t + \tilde{C}_1$$

$$\rho \cdot \sqrt{\frac{g}{\rho}} = \sqrt{\rho}\cdot\sqrt{g} = \sqrt{\rho g}$$

$$\frac{v}{\sqrt{\frac{g}{\rho}}} = \sqrt{\frac{\rho}{g}}\, v$$

$v(0) = v_0$: initial velocity

$$\arctan\left(\sqrt{\frac{\rho}{g}}\, \underbrace{v(0)}_{v_0}\right) = -\cancel{\sqrt{\rho g}\cdot 0}^{\,0} + \tilde{C}_1 \Rightarrow \tilde{C}_1 = \arctan\left(\sqrt{\frac{\rho}{g}}\cdot v_0\right)$$

$$\therefore \quad \arctan\sqrt{\frac{\rho}{g}}\, v = -\sqrt{\rho g}\, t + \arctan\sqrt{\frac{\rho}{g}}\, v_0$$

$$\tan(\ldots) \Rightarrow \quad \sqrt{\frac{\rho}{g}}\, v = \tan\left(-\sqrt{\rho g}\, t + \underbrace{\arctan\sqrt{\frac{\rho}{g}}\, v_0}_{A}\right)$$

or $$\boxed{v(t) = \sqrt{\frac{g}{\rho}}\tan\left(-\sqrt{\rho g}\cdot t + A\right)} \qquad A = \arctan\sqrt{\frac{\rho}{g}}\, v_0$$

$$y(t) = \int v(t)\, dt$$

<u>Note</u> $$\int \tan u\, du = \int \frac{\sin u}{\cos u}\, du = \left| \begin{array}{l} U = \cos u \\ dU = -\sin u\, du \end{array} \right| =$$

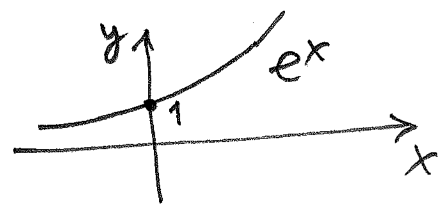
$$= -\int \frac{dU}{U} = -\ln|U| + C = -\ln|\cos u| + C$$

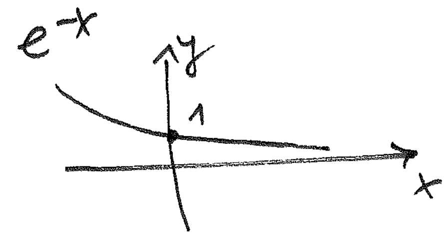
. . .

$$\boxed{y(t) = y_0 + \frac{1}{\rho} \ln \left| \frac{\cos(A - \sqrt{\rho g}\, t)}{\cos A} \right| , \quad \text{where } y_0 = y(0)}$$

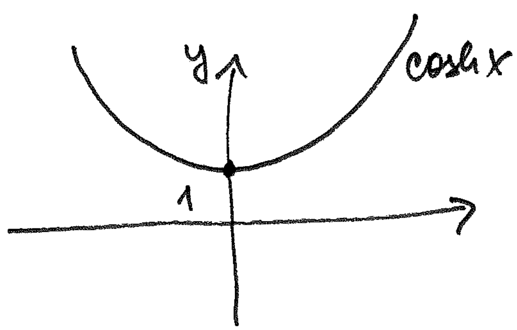
Recall hyperbolic functions

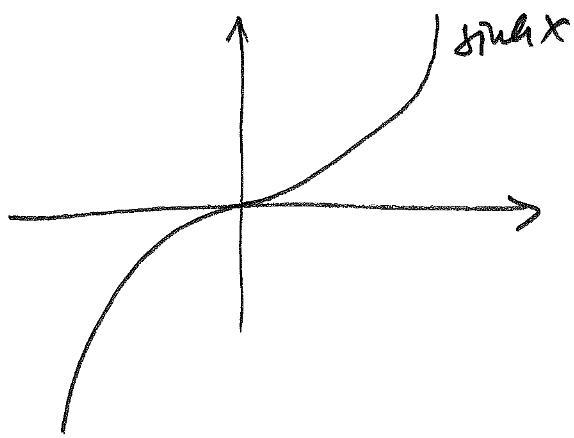
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

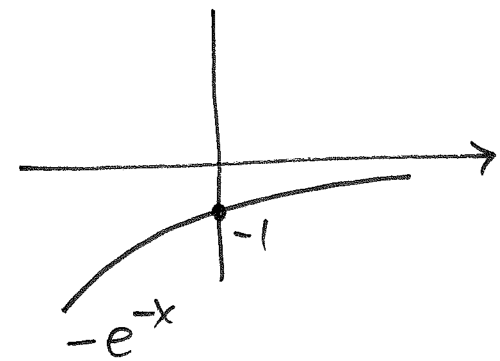




$$\tanh x = \frac{\sinh x}{\cosh x}$$





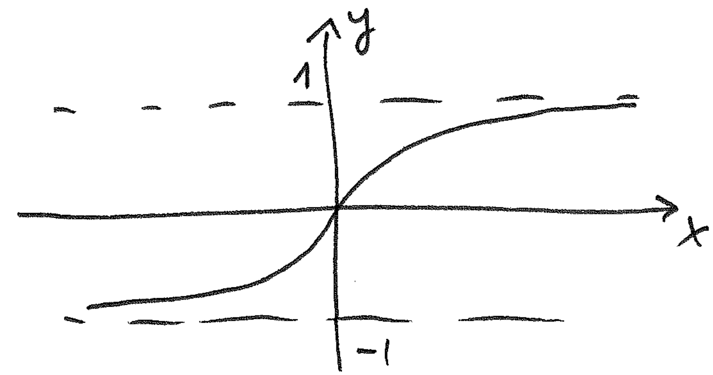


$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - \overset{\sim 0}{e^{-x}}}{e^x + \underset{\sim 0}{e^{-x}}} = 1$$

$$\lim_{x\to -\infty} \tanh x = \lim_{x\to\infty} \frac{\overset{\sim 0}{e^x} - e^{-x}}{\underset{\sim 0}{e^x} + e^{-x}} = -1$$



Downward motion: $v<0 \Rightarrow |v| = -v$

$$\frac{dv}{dt} = -g + \rho v^2$$ sign difference w/ upward motion

$$\frac{dv}{dt} = -g\left(1 - \frac{\rho}{g} v^2\right)$$: separable DE

$$\frac{dv}{1-\frac{\rho}{g}v^2} = -g\,dt$$

<u>Substitution:</u> $u = \sqrt{\frac{\rho}{g}}\,v$

$$\int \frac{du}{1-u^2} = \operatorname{arctanh} u + C$$

One can solve for $v(t)$:

$$\boxed{v(t) = \sqrt{\frac{g}{\rho}}\tanh\left(B - \sqrt{\rho g}\,t\right), \qquad B = \operatorname{arctanh}\left(v_0\sqrt{\frac{\rho}{g}}\right)}$$

<u>Q</u> Do we have a limiting velocity?

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\sqrt{\frac{g}{\rho}}\underbrace{\tanh(\underbrace{B-\sqrt{\rho g}\,t}_{\to -\infty})}_{\to -1} = -\sqrt{\frac{g}{\rho}} : \text{\underline{limiting velocity}}$$