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Lecture 12

<u>Downward motion:</u> $v < 0$ (Cont'd)

$$\frac{dv}{dt} = -g + \rho v^2$$

We found $v(t) = \sqrt{\frac{g}{\rho}} \tanh\left(B - \sqrt{\rho g}\, t\right)$, $B = \operatorname{arctanh}\left(v_0 \sqrt{\frac{\rho}{g}}\right)$

$$\lim_{t \to \infty} v(t) = \underbrace{-\sqrt{\frac{g}{\rho}}}_{= v_\tau} : \text{ limiting velocity}$$

$\rho = 0.0011$

Terminal speed $|v_\tau| = \sqrt{\frac{g}{\rho}}$

Compare this w/ $v_\tau = -\frac{g}{\rho}$ when $F_R = -kv$, $\rho = 0.04$

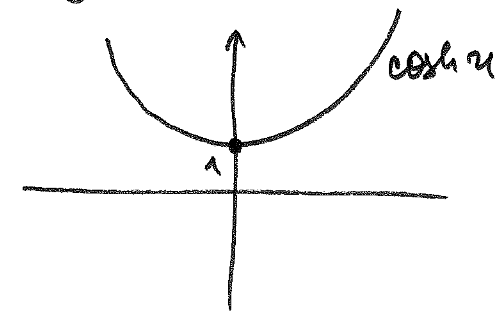
To get displacement $y(t)$ we integrate $v(t)$

$$y(t) = \int v(t)\,dt = \ldots = y_0 - \frac{1}{\rho} \ln \left| \frac{\cosh\left(B - t\sqrt{\rho g}\right)}{\cosh B} \right|, \quad y_0 = y(0)$$

To derive $y(t)$, we used

$$\int \tanh u \, du = \int \frac{\sinh u}{\cosh u} \, du = \left| \begin{matrix} U = \cosh u \\ dU = \sinh u \end{matrix} \right| = \int \frac{dU}{U} = \ln|U| + C =$$

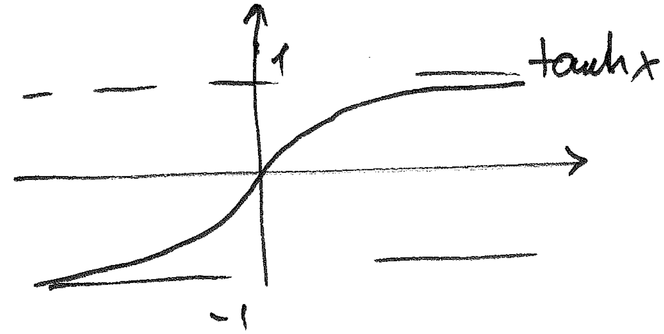
$$= \ln \cosh u + C$$



$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

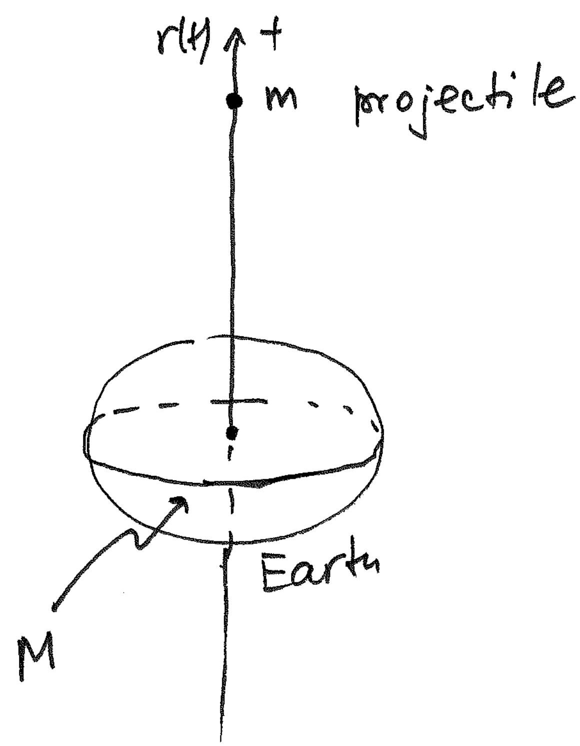
$x \gg 1$ $(x \to \infty)$ $\Rightarrow \tanh x \sim \dfrac{e^x}{e^x} = 1$

$x \to -\infty$ $\Rightarrow \tanh x \sim \dfrac{-e^{-x}}{e^{-x}} = -1$



Escape Velocity

Problem: What is the initial velocity v_0 necessary for a projectile fired from the surface of Earth to escape Earth?



$r(t)$: distance between the center of Earth and projectile

R: radius of Earth

m: mass of projectile

M: mass of Earth

$F = \dfrac{GMm}{r^2}$: gravitational force of attraction

r: distance between two masses m & M

$G = 6.6726 \times 10^{11} \; N\left(\frac{m}{kg}\right)^2$: universal gravitational const

$ma = F, \quad a = \dfrac{dv}{dt}$

$m\dfrac{dv}{dt} = -\dfrac{GMm}{r^2} \quad \Big| \; \dfrac{1}{m}$

$$\frac{dv}{dt} = -\frac{GM}{r^2} \qquad v = v(t), \quad r = r(t)$$

only one DE!

$$\frac{dv}{dt} \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{dv}{dr} \cdot \underbrace{\frac{dr}{dt}}_{v(t)} = v\frac{dv}{dr}$$

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$: DE for $v = v(r)$, 1st order, separable

$$v\,dv = -\frac{GM}{r^2}\,dr$$

$$\int v\,dv = -GM\int \frac{dr}{r^2}$$

$$\frac{v^2}{2} = GM \cdot \frac{1}{r} + \tilde{C} \quad \Big|\cdot 2$$

$$v^2 = \frac{2GM}{r} + C$$

ICs: at $t=0$, $r(0) = R$, $v(0) = v_0$ – need to find

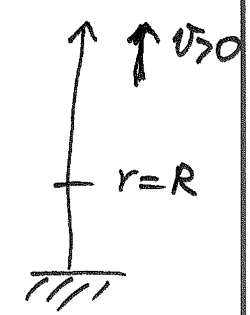
$$v^2 = \frac{2GM}{r} + C \qquad \text{or} \qquad v(R) = v_0$$

at $t=0$: $\quad v_0^2 = \frac{2GM}{R} + C \Rightarrow C = v_0^2 - \frac{2GM}{R}$

$$\therefore \quad v^2 = \frac{2GM}{r} + \left(v_0^2 - \frac{2GM}{R}\right) = \frac{2GM}{r} + v_0^2 - \frac{2GM}{R}$$

To find escape velocity, we need to impose

$v>0$ for all t



$$v^2 = \underbrace{\frac{2GM}{r}}_{>0} + v_0^2 - \frac{2GM}{R} > v_0^2 - \frac{2GM}{R}$$

if we require $v_0^2 - \frac{2GM}{R} > 0 \Rightarrow v^2 > 0$ automatically for all t

$v_0 = \sqrt{\frac{2GM}{R}}$: escape velocity

On Earth,

$$M = 5.975 \times 10^{24} \text{ kg}, \qquad R = 6.378 \times 10^{6} \text{ m}$$

$$\Rightarrow v_0 \approx 11,180 \text{ m/s}$$

On Moon,

$$M = 7.35 \times 10^{22} \text{ kg}, \qquad R = 1.74 \times 10^{6} \text{ m}$$

$$\Rightarrow v_0 \approx 2,375 \text{ m/s}$$

2.4 Numerical Methods for Solving ODEs

Consider 1st order DE

$$\boxed{\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0} \qquad (1)$$

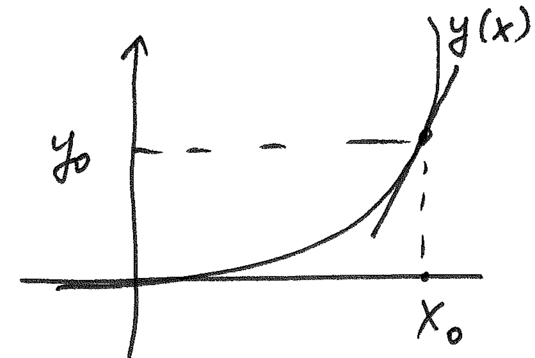
Recall Taylor series for $y(x)$ about $x = x_0$:

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \dots$$

Let $h = x - x_0$, $\quad x = x_0 + h$

$$\underbrace{y(x_0 + h) = y(x_0) + y'(x_0)\,h}\, + \frac{y''(x_0)}{2!} h^2 + \frac{y'''(x_0)}{3!} h^3 + \dots$$

Note: the more terms we keep in this sum, the better approximation of $y(x_0 + h)$ we will get.



Note $y(x_0) = y_0$

$$y'(x_0) = \frac{dy}{dx}(x_0) = \left.\frac{dy}{dx}\right|_{x = x_0} = f(x_0, y_0) \quad \text{from (1)}$$

$$y''(x_0) = \left.\frac{d}{dx}\Big(\underbrace{\frac{dy}{dx}}_{f(x,y)}\Big)\right|_{x = x_0} = \left.\frac{d}{dx}\big(f(x,y)\big)\right|_{x = x_0} = \left.\Big(f_x \cdot \underbrace{\frac{dx}{dx}}_{1} + f_y \cdot \underbrace{\frac{dy}{dx}}_{f}\Big)\right|_{x = x_0} =$$

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$$= f_x(x_0,y_0) + f_y(x_0,y_0)\cdot f(x_0,y_0)$$

etc.

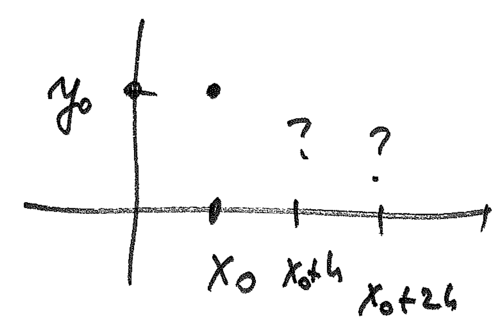
We will use only the first two terms in Taylor series:

$$y(x_0+h) \approx \underbrace{y(x_0)}_{= y_0} + \underbrace{y'(x_0)}_{\frac{dy}{dx}(x_0) = f(x_0,y_0)} h$$

$$\Rightarrow \boxed{y(x_0+h) \approx y_0 + f(x_0,y_0)\cdot h}$$ two-term approximation

Q Can we use two-term approximation to compute (approximately)

$y(x_0+h)$, $y(x_0+2h)$, $y(x_0+3h)$, …, $y(x_0+nh)$



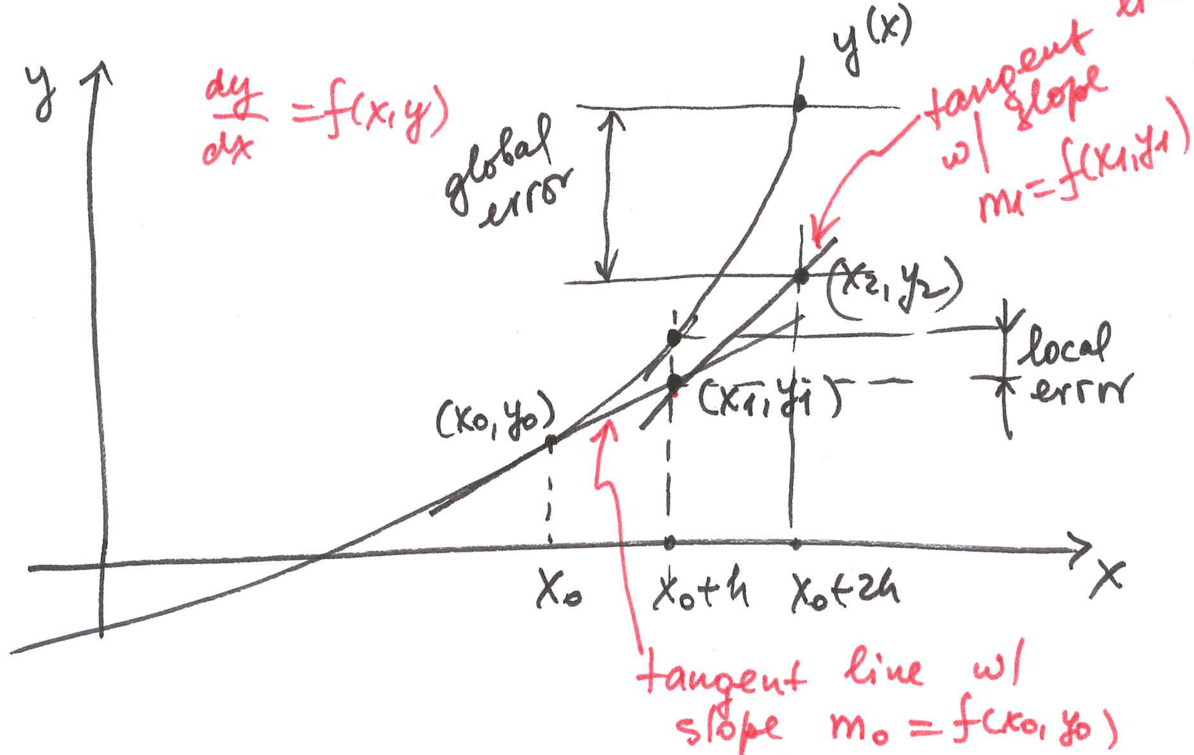
Eule's method (Constant Slope Method)

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

We are given x_0, y_0. Then

$$y(x_0+h) \approx y_0 + f(x_0, y_0)\,h$$

Note $y_0 + f(x_0, y_0)\,h$ is not the same as $y(x_0+h)$. This is only an approximation. Since we kept only first two terms in Taylor series, the first term that we neglected was proportional to h^2. Hence, the error at every step is $K \cdot h^2$, where K is some constant. This error is called local error (error over one step)



Let $x_1 = x_0 + h$

$$y_1 = y_0 + f(x_0, y_0)\, h$$

Then

$$x_2 = x_1 + h$$

$$y_2 = y_1 + f(x_1, y_1)\, h$$

Repeat the procedure.

Since the error at every step is Kh^2 (local error), the method gives better results if h is small, but if h is too small, then roundoff error dominates.