Lecture 13

#27 §1.5 HW #3

$$(x + y e^{y}) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \Rightarrow x + y e^{y} = \frac{dx}{dy}$$

or $\frac{dx}{dy} - x = y e^{y}$, $x = x(y)$

$$\frac{dx}{dy} + P(y)x = Q(y)$$

$P(y) = -1$, $Q(y) = y e^{y}$

$$\rho(y) = e^{\int P(y)\,dy} = e^{-y}$$

$$\boxed{\rho x = \int \rho Q\,dy + C}$$

$$\rho \cdot x = \int \rho \cdot Q\,dy + C$$

$$e^{-y} \cdot x = \int \cancel{e^{-y}} \cdot (y) \cancel{e^{y}}\,dy + C$$

$$e^{-y} \cdot x = \frac{y^2}{2} + C \quad \Big| \cdot e^{y}$$

$$x(y) = e^{y}\left(\frac{y^2}{2} + C\right)$$

#19 §2.3 HW #4

A motorboat starts from rest ($v(0)=v_0=0$). Its motor provides a constant acceleration of $4\ ft/s^2$, but water resistance causes a deceleration of $v^2/400\ ft/s^2$. Find v when $t=10\,s$, and also find the limiting velocity as $t\to\infty$ (ie. max possible speed of the boat).

Solution

$$\frac{dv}{dt} = 4 - \frac{v^2}{400}$$

acceleration

separable DE

$$\frac{dv}{4-\frac{v^2}{400}} = dt \qquad 4\cdot 4\cdot 100 = 4\cdot 400 = 4^2\cdot 10^2 = 40^2$$

$$\frac{400}{40^2-v^2}\,dv = dt$$

$$\left(\frac{5}{40-v}+\frac{5}{40+v}\right)dv = dt$$

$$5\left(-\ln|40-v| + \ln|40+v|\right) = t+\tilde{C}$$

$$\exp \Big/ \ \ln\left|\frac{40+v}{40-v}\right| = \frac{t}{5}+\tilde{\tilde{C}} \qquad \ln\frac{a}{b} = \ln a - \ln b$$

Limiting velocity: $\lim_{t\to\infty} v(t)$: equilibrium solution

$$\frac{dv}{dt}=0 \Rightarrow 4-\frac{v^2}{400}=0 \Rightarrow v^2 = 4\cdot 400 \Rightarrow v_{lim} = 40\ ft/s$$

$$\frac{400}{40^2-v^2} = \frac{A}{40-v}^{(40+v)} + \frac{B}{40+v}^{(40-v)} = \frac{5}{40-v}+\frac{5}{40+v}$$

$$400 = A(40+v)+B(40-v)$$

v^1: $0 = A-B \Rightarrow B=A$

v^0: $400 = 40(A+B) \Rightarrow A+B=10$

$2A=10 \Rightarrow A=5=B$

evaluate $v(10)=\dots$

$$\Rightarrow \frac{40+v}{40-v} = Ce^{\frac{t}{5}} \qquad v(0)=0 \Rightarrow C=1$$

$$40+v = \underset{=1}{C}e^{t/5}\cdot(40-v)$$

$$v\left[1+e^{t/5}\right] = 40e^{t/5}-40 \Rightarrow \boxed{v(t) = \frac{40\left[e^{t/5}-1\right]}{1+e^{t/5}}}$$

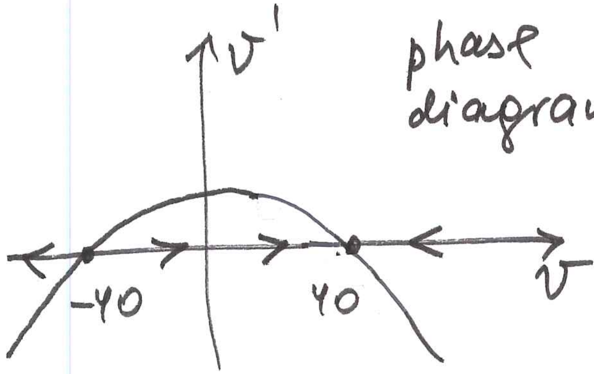
$$\lim_{t\to\infty} v(t) = 40 \ \checkmark$$

$$\frac{dv}{dt} = 4 - \frac{v^2}{400}$$

$$\frac{dv}{dt} = \frac{40^2 - v^2}{400}$$

$$\frac{dv}{dt} = \frac{(40-v)(40+v)}{400}$$

$$\frac{dv}{dt} = -\frac{(v-40)(v+40)}{400}$$

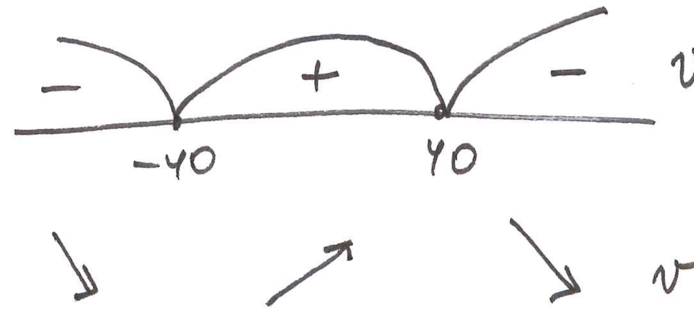


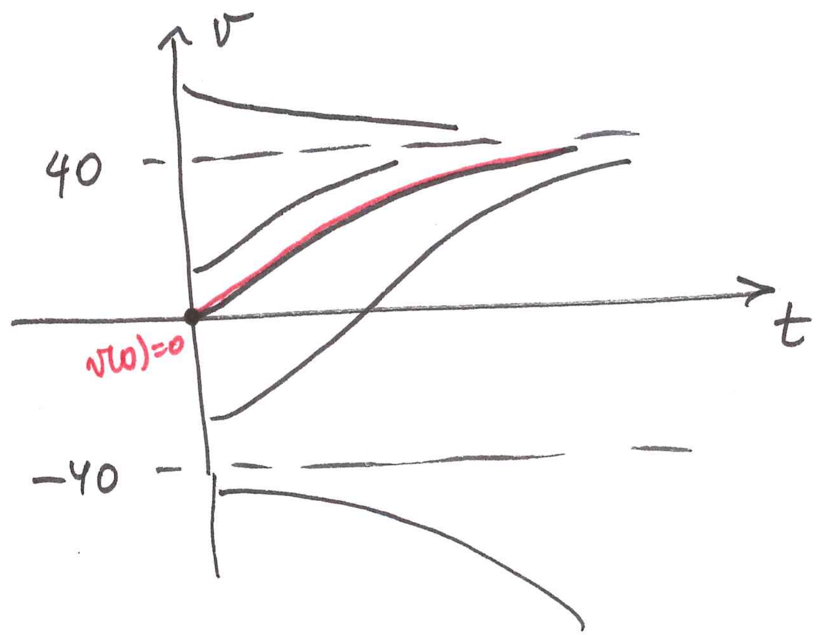
phase diagram

$$P(x) = a(x-x_1)(x-x_2)\dots(x-x_n)$$

$a>0 \Rightarrow +$ on very right subinterval

$a<0 \Rightarrow -$



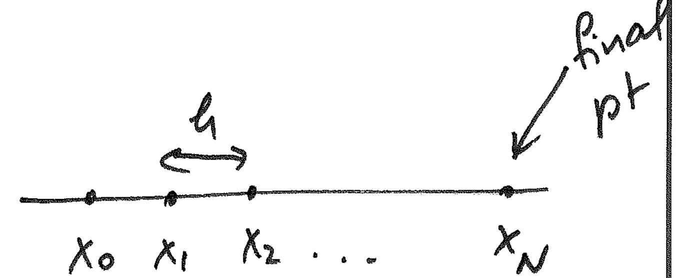


Euler's method (Cont'd)

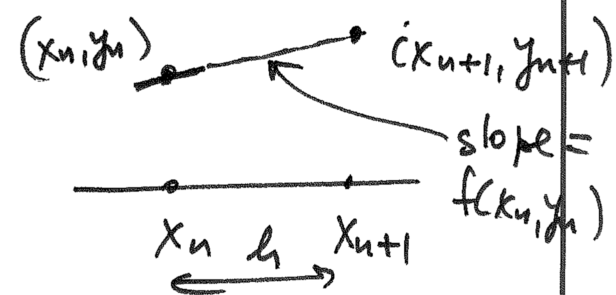
For Euler's method, the <u>global error</u> (error at the final time or <u>cumulative error</u>) is $C \cdot h$. This implies that Euler's method is 1st order accurate, i.e. error $= O(h)$ (of order of h). If you decrease h by a half, i.e. $h \to \frac{h}{2}$, the error will decrease by approximately half as well.

algorithm for Euler's method to solve IVP

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$



Initialize x_0, y_0, h, N (# of steps)
Loop from $n=0$ to $N-1$: $\quad x_{n+1} = x_n + h$
$\quad y_{n+1} = y_n + h \cdot f(x_n, y_n)$



Ex Solve approximately

$$\frac{dy}{dx} = \frac{x^2 + xy}{x + 2y}, \qquad \underset{= x_0}{y(1)} = \overset{= y_0}{2}$$

$$\underset{= x_0}{1} \quad x_1 \quad x_2 \quad x_3 \quad \underset{= x_4}{2}$$

Find $y(2)$ in four steps.

$$x_0 = 1, \quad y_0 = 2, \quad N = 4, \quad h = \frac{2-1}{N} = \frac{1}{4} = 0.25$$

$$f(x,y) = \frac{x^2 + xy}{x + 2y} \qquad f(x_n, y_n) = \frac{x_n^2 + x_n \cdot y_n}{x_n + 2y_n}$$

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \cdot f(x_n, y_n) = y_n + h \cdot \underbrace{\dfrac{x_n^2 + x_n y_n}{x_n + 2y_n}}_{m_n} \end{cases}$$

$x_0 = 1, \; y_0 = 2$

$n = 0$

$n+1 = 0+1 = 1$

$$\left| \begin{array}{l} x_1 = x_0 + h = 1 + 0.25 = \boxed{1.250} \\ y_1 = y_0 + h \cdot \overbrace{\dfrac{x_0^2 + x_0 y_0}{x_0 + 2y_0}}^{= m_0} = 2 + 0.25 \cdot \dfrac{1^2 + 1 \cdot 2}{1 + 2 \cdot 2} = 2 + 0.25 \cdot \overset{m_0}{(0.6)} = \boxed{2.150} \end{array} \right.$$

$\underline{\underline{n=1}}$

$$x_2 = x_1 + h = 1.250 + 0.25 = \boxed{1.5}$$

$$y_2 = y_1 + h \underbrace{\frac{x_1^2 + x_1 y_1}{x_1 + 2y_1}}_{m_1} = 2.150 + 0.25 \cdot \frac{1.250^2 + (1.250)\cdot(2.150)}{1.250 + 2\cdot(2.150)} =$$

$$= 2.150 + 0.25 \cdot \underbrace{(0.765)}_{m_1} = \boxed{2.341}$$

$n=2$

$$x_3 = x_2 + h = 1.5 + 0.25 = \boxed{1.75}$$

$$y_3 = y_2 + h \underbrace{\frac{x_2^2 + x_2 y_2}{x_2 + 2y_2}}_{m_2} = 2.341 + 0.25 \cdot \underbrace{\frac{1.500^2 + (1.500)(2.341)}{1.500 + 2(2.341)}}_{m_2} =$$

$$= 2.341 + 0.25 \underbrace{(0.931)}_{m_2} = \boxed{2.573}$$

$n=3$

$$x_4 = x_3 + h = 1.75 + 0.25 = \boxed{2.000}$$

$$y_4 = y_3 + h \, \underbrace{\frac{x_3^2 + x_3 y_3}{x_3 + 2y_3}}_{m_3} = 2.573 + 0.25 \cdot \frac{1.750^2 + (1.750)(2.573)}{1.750 + 2(2.573)} =$$

$$= 2.573 + 0.25 \underbrace{(1.097)}_{m_3} = \boxed{2.847}$$

$$\Rightarrow \boxed{y(2) \approx 2.847}$$