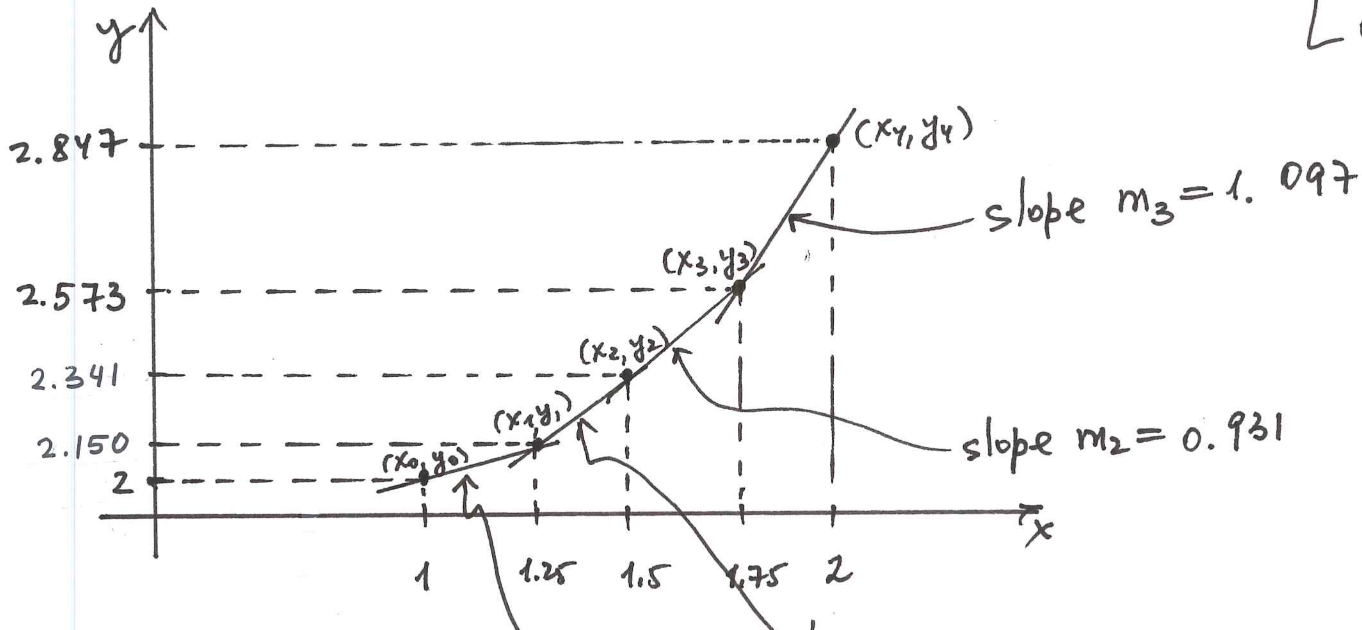
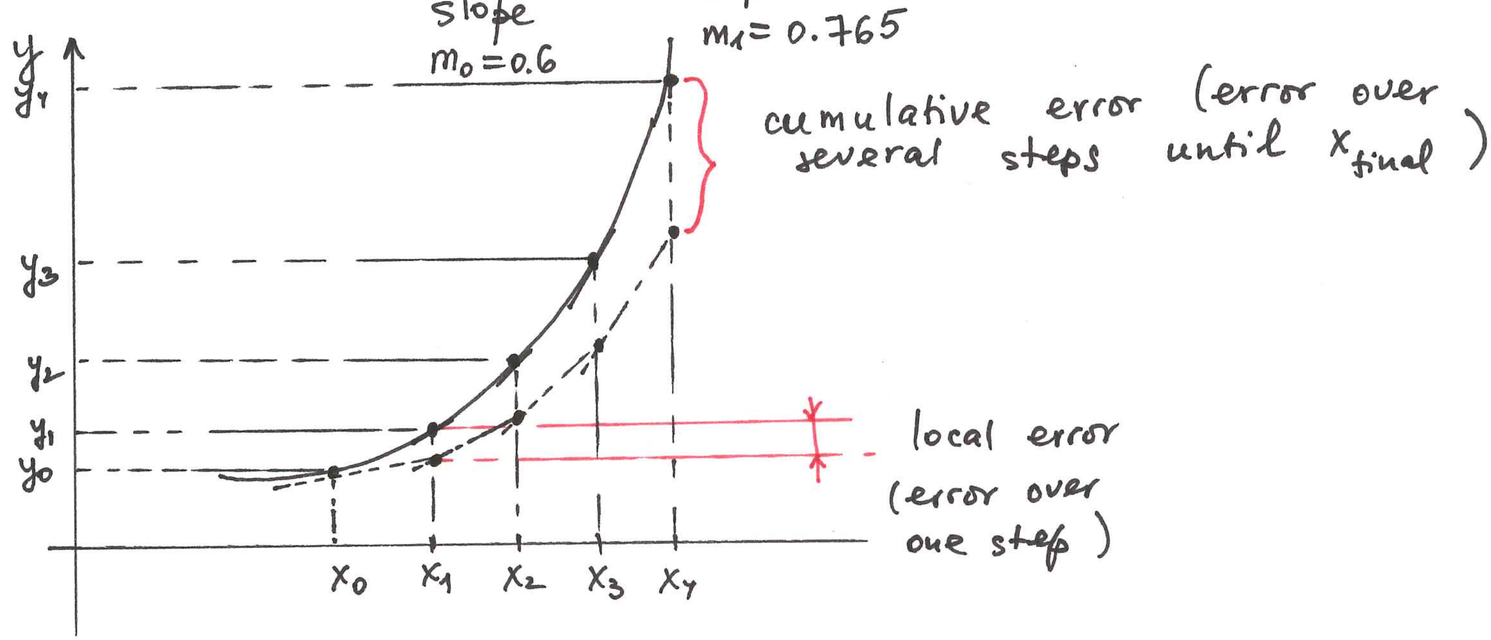
Lecture 14



y
x
2.847
2.573
2.341
2.150
2
1
1.25
1.5
1.75
2
(x_0, y_0)
(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4)
slope $m_3 = 1.097$
slope $m_2 = 0.931$
slope $m_1 = 0.765$
slope $m_0 = 0.6$



y
y_4
y_3
y_2
y_1
y_0
x_0
x_1
x_2
x_3
x_4
cumulative error (error over several steps until x_{final})
local error (error over one step)

Note: cumulative error $\nearrow$ as you go from x_0

Cum. error $\sim \underset{\text{const}}{C}\, h$

$h \to \frac{h}{2} \Rightarrow$ error $\approx \frac{1}{2}$ error

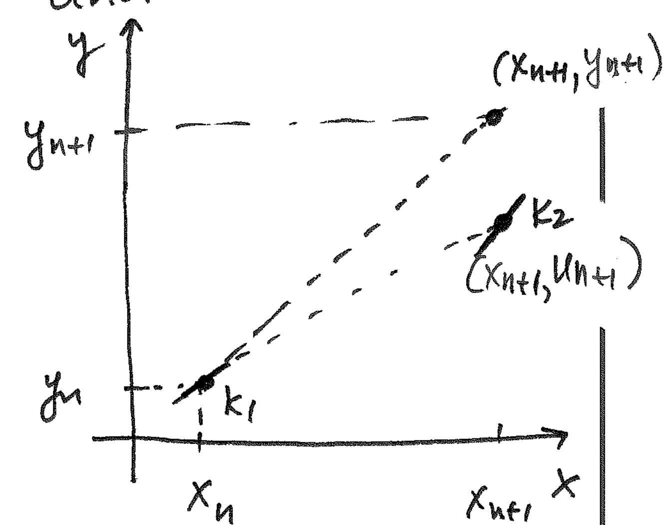
In Euler's method we use only one point (x_n, y_n) to go to the next point (x_{n+1}, y_{n+1}). Would it be better if we also use the slope at (x_{n+1}, y_{n+1})? Yes, but we don't know the value y_{n+1}. We estimate (predict) y_{n+1} and correct it.

$\frac{dy}{dx} = f(x,y)$

Modified Euler's Method

$$u_{n+1} = y_n + h\, f(x_n, y_n) : \quad \text{predictor}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, u_{n+1}) \right] \quad \text{corrector}$$

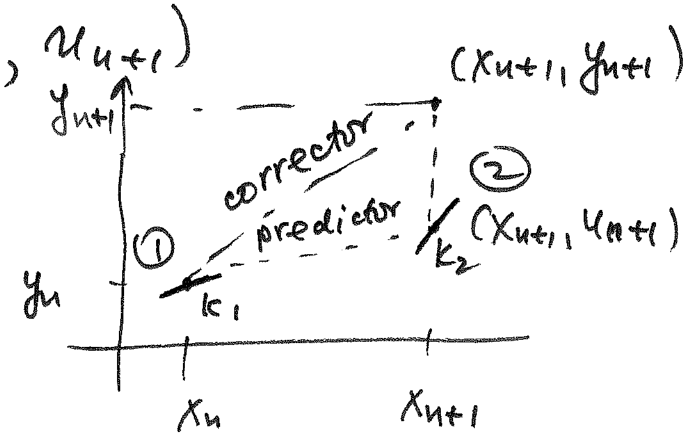


or $k_1 = f(x_n, y_n)$: slope of tangent line at (x_n, y_n)

$u_{n+1} = u_n + h \cdot k_1$

$k_2 = f(x_{n+1}, u_{n+1})$: slope at (x_{n+1}, u_{n+1})

$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$



another form

predictor:

$u_{n+1} = y_n + h\, f(x_n, y_n)$

corrector:

$$y_{n+1} = y_n + \frac{h}{2}\Big[f(x_n, y_n) + f(\underbrace{x_n + h}_{x_{n+1}}, \underbrace{y_n + h f(x_n, y_n)}_{u_{n+1}})\Big]$$

Note The modified Euler's method matches first three terms of Taylor series of $y(x_0+h)$.

$$y(x_0+h) = \underbrace{y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2}_{} + \frac{y'''(x_0)}{3!}h^3 + \dots$$

Note: Since the modified Euler's method matches the first three terms of Taylor series (up to and including term w/ h^2), the local error is proportional to h^3, while the cumulative error (global error) is proportional to h^2. This implies that when h is cut by half, the error (cumulative error) will decrease approximately by $\frac{1}{4}$.

$$\text{Error} \sim C h^2 \quad : \quad \underline{\text{modified Euler}}$$

$$h \to \frac{h}{2} \Rightarrow C\left(\frac{h}{2}\right)^2 = \frac{1}{4} C h^2, \text{ i.e. } \text{Error} \to \frac{1}{4}\text{Error}$$

Recall: for Euler method

$$E \sim C h$$

$$h \to \frac{h}{2} \Rightarrow \text{Error} \to \frac{1}{2}\text{Error}$$

Modified Euler's method for solving

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

Initialize x_0, y_0, h, N.

$$x_{n+1} = x_n + h$$

$$u_{n+1} = y_n + h\, f(x_n, y_n) \quad : \quad \text{predictor}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, u_{n+1}) \right] \quad : \quad \text{corrector}$$

or use

$$x_{n+1} = x_n + h$$

$$k_1 = f(x_n, y_n)$$

$$u_{n+1} = y_n + h \cdot k_1$$

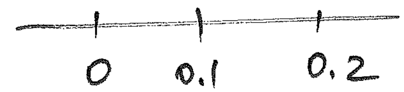
$$k_2 = f(x_{n+1}, u_{n+1})$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

Ex Use the modified Euler method to solve

$$y' = (x+y-1)^2, \qquad y(0) = 2$$

Find $y(0.2)$ in 2 steps.



$$x_0 = 0, \quad y_0 = 2, \quad N = 2, \qquad h = \frac{0.2-0}{N} = 0.1$$

$$f(x,y) = (x+y-1)^2$$

$$x_{n+1} = x_n + h$$

$$u_{n+1} = y_n + h\,(x_n + y_n - 1)^2$$

$$y_{n+1} = y_n + \frac{h}{2}\left[(x_n + y_n - 1)^2 + (x_{n+1} + u_{n+1} - 1)^2 \right]$$

$$x_1 = x_0 + h = 0 + 0.1 = \boxed{0.1}$$

$$u_1 = y_0 + h\,(x_0 + y_0 - 1)^2 = 2 + (0.1)(0 + 2 - 1)^2 = \boxed{2.1}$$

$$y_1 = y_0 + \frac{h}{2}\left[(x_0 + y_0 - 1)^2 + (x_1 + u_1 - 1)^2 \right] =$$

University of Idaho

$$= 2 + \frac{0.1}{2}\left[(0+2-1)^2 + (0.1+2.1-1)^2\right] = \boxed{2.122}$$

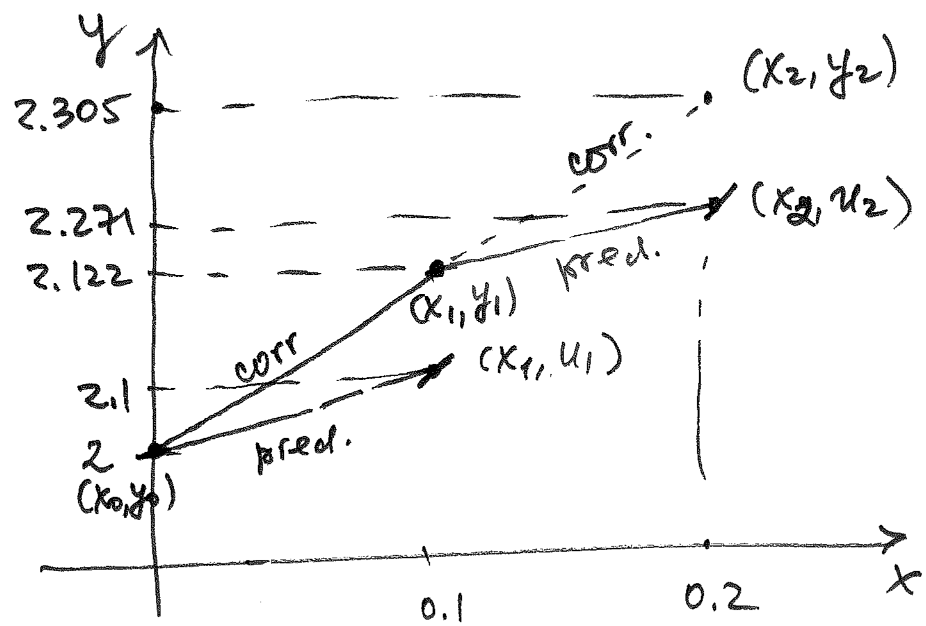
$$x_2 = x_1 + h = 0.1 + 0.1 = \boxed{0.2}$$

$$u_2 = y_1 + h(x_1 + y_1 - 1)^2 = 2.122 + 0.1(0.1 + 2.122 - 1)^2 = 2.271$$

$$y_2 = y_1 + \frac{h}{2}\left[(x_1 + y_1 - 1)^2 + (x_2 + u_2 - 1)^2\right] =$$

$$= 2.122 + \frac{0.1}{2}\left[(0.1 + 2.122 - 1)^2 + (0.2 + 2.271 - 1)^2\right] = \boxed{2.305}$$

Hence, $\boxed{y(0.2) \approx 2.305}$



4^{th} order Runge-Kutta method

We saw when we averaged the slopes in the interval $[x_n, x_{n+1}]$ by taking $\frac{1}{2}$ the slope at (x_n, y_n) and $\frac{1}{2}$ the slope at (x_{n+1}, y_{n+1}), we obtained the scheme (modified Euler) that matched the first three terms in Taylor expansion of $y(x_0+h)$ (up to h^2 term).

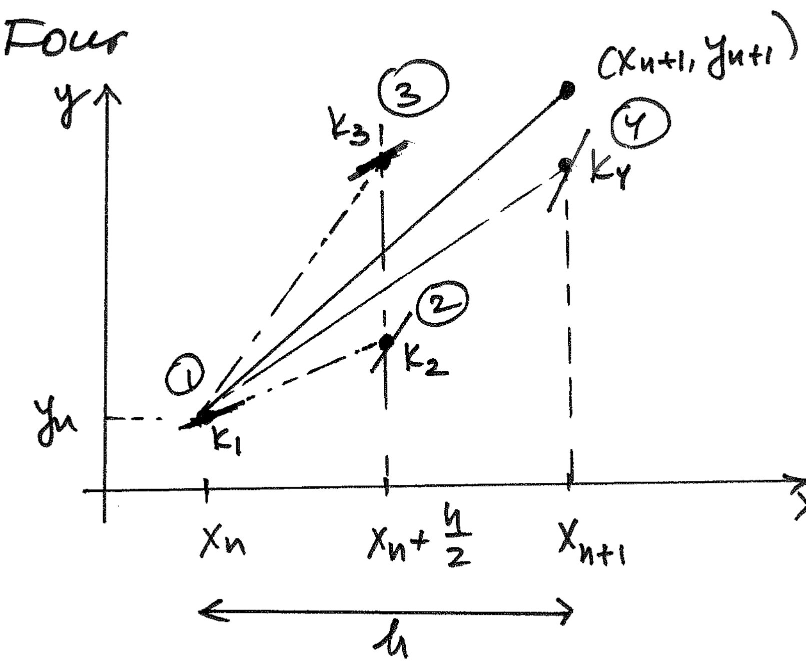
Q Can we use more points in $[x_n, x_{n+1}]$ to match Taylor series up to and including h^4 term? Yes

Q How many points do we use? Four

Q Where are they located?

at x_n, $x_n+\frac{h}{2}$, $x_n+\frac{h}{2}$, $x_{n+1}=x_n+h$

Q Do we give the equal weight to these points? No, we give twice the weight to points at $x_n+\frac{h}{2}$.



$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$$

$$k_4 = f(\underbrace{x_n + h}_{x_{n+1}}, y_n + h k_3)$$

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$