University of Idaho

Lecture 15

4^{th} order Runge-Kutta method (Cont'd)

$$y' = f(x, y), \quad y(x_0) = y_0$$

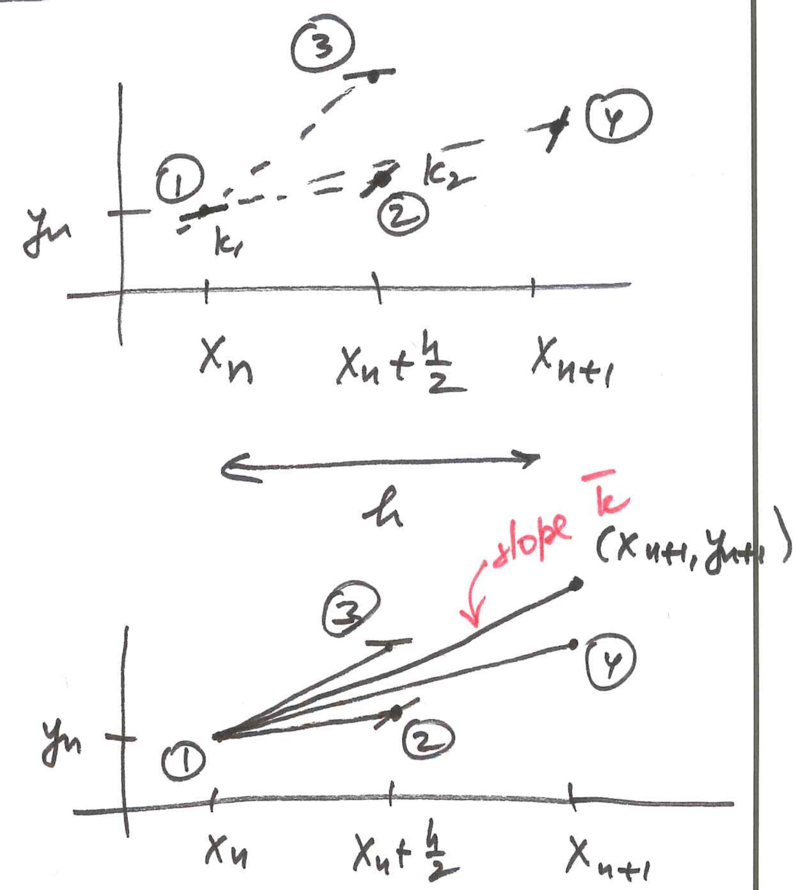
$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + k_1 \cdot \frac{h}{2}\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + k_2 \cdot \frac{h}{2}\right)$$

$$k_4 = f(x_n + h, y_n + k_3 \cdot h)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$



or $\bar{k} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$: average slope

$$y_{n+1} = y_n + h \cdot \bar{k}$$

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

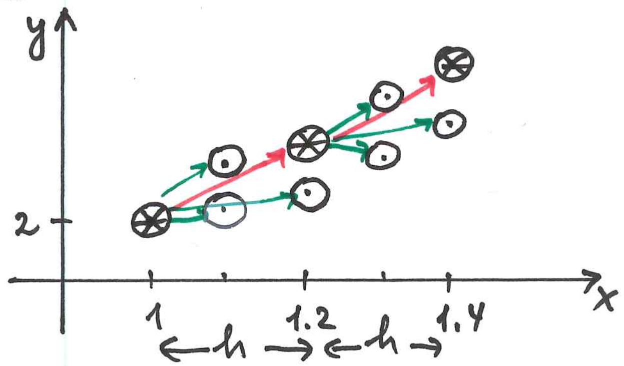
global (cumulative error) is of order $O(h^4)$: if we decrease h by a half, the error will decrease by $\frac{1}{16}$

$$\text{Error} = C h^4$$

$$h \to \frac{h}{2} \qquad \underbrace{C\left(\frac{h}{2}\right)^4 = \frac{1}{2^4}\underbrace{Ch^4}_{\text{error w/ } h}}_{\text{error w/ step } \frac{h}{2}}$$

Ex $\frac{dy}{dx} = x + \sqrt{y}$, $\quad y(1) = 2$

Find $y(1.4)$ in 2 steps using the 4^{th} order Runge-Kutta method.



$h = 0.2$, $N = 2$

$k_1 = f(x_n, y_n)$

$k_2 = f(x_n + \frac{h}{2}, y_n + k_1 \cdot \frac{h}{2})$

$k_3 = f(x_n + \frac{h}{2}, y_n + k_2 \cdot \frac{h}{2})$

$k_4 = f(x_n + h, y_n + k_3 \cdot h)$

$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$f(x, y) = x + \sqrt{y}$

$f(x_n, y_n) = x_n + \sqrt{y_n}$

at (x_0, y_0) $\quad x_0 = 1$, $y_0 = 2$

$k_1 = f(x_0, y_0) = x_0 + \sqrt{y_0} = 1 + \sqrt{2} = 2.4142$

$k_2 = f(x_0 + \frac{h}{2}, y_0 + k_1 \cdot \frac{h}{2}) = x_0 + \frac{h}{2} + \sqrt{y_0 + k_1 \cdot \frac{h}{2}} = 1 + \frac{0.2}{2} + \sqrt{2 + 2.4142 \cdot \frac{0.2}{2}} =$
$= 2.5971$

$k_3 = f(x_0 + \frac{h}{2}, y_0 + k_2 \cdot \frac{h}{2}) = x_0 + \frac{h}{2} + \sqrt{y_0 + k_2 \cdot \frac{h}{2}} =$

$= 1 + \frac{0.2}{2} + \sqrt{2 + 2.5971 \cdot \frac{0.2}{2}} = 2.6032$

$k_4 = f(x_0 + h, y_0 + k_3 \cdot h) = x_0 + h + \sqrt{y_0 + k_3 \cdot h} = 1 + 0.2 + \sqrt{2 + 2.6032 \cdot 0.2} = 2.7876$

$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2 + \frac{0.2}{6}(2.4142 + 2(2.5971) + 2 \cdot (2.6032) +$

$+ 2.7876) = \boxed{2.5201}$

$x_1 = x_0 + h = \boxed{1.2}$

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at (x_1, y_1) $\quad x_1 = 1.2000, \quad y_1 = 2.5201$

$$k_1 = f(x_1, y_1) = 1.2 + \sqrt{y_1} = 1.2 + \sqrt{2.5201} = 2.7875$$

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + k_1 \cdot \frac{h}{2}\right) = x_1 + \frac{h}{2} + \sqrt{y_1 + k_1 \cdot \frac{h}{2}} = 1.2 + \frac{0.2}{2} + \sqrt{2.5201 + 2.7875 \cdot \frac{0.2}{2}} =$$

$$= 2.9730$$

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + k_2 \cdot \frac{h}{2}\right) = x_1 + \frac{h}{2} + \sqrt{y_1 + k_2 \cdot \frac{h}{2}} = 1.2 + \frac{0.2}{2} + \sqrt{2.5201 + 2.9730 \cdot \frac{0.2}{2}} =$$

$$= 2.9785$$

$$k_4 = f(x_1 + h, y_1 + k_3 \cdot h) = x_1 + h + \sqrt{y_1 + k_3 \cdot h} = 1.2 + 0.2 + \sqrt{2.5201 + 2.9785 \cdot (0.2)}$$

$$= 3.1652$$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.5201 + \frac{0.2}{6}(2.7875 + 2(2.9730) +$$

$$+ 2(2.9785) + 3.1652) = 2.5201 + 0.2\,\underbrace{(2.9760)}_{\text{aver. slope}}$$

$$= \boxed{3.1153}$$

$x_2 = x_1 + h = 1.4$

$\Rightarrow$ at (x_2, y_2) $\quad x_2 = 1.4000, \quad y_2 = 3.1153 \quad \Rightarrow \boxed{y(1.4) \approx 3.1153}$

Ch. 3 Linear Equations of Higher Order

Def a linear n^{th} order DE is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} +$$

$$+ a_0(x)\, y = R(x), \qquad a_n \neq 0$$

$y = ax + b$

If $a_i(x)$, $i = 0, \dots, n$ are constants, then DE is a linear DE w/ const coefficients. Otherwise, this DE is a linear DE w/ variable coefficients.

If $R(x) \equiv 0$, then linear DE is called homogeneous. Otherwise, linear DE is nonhomogeneous.

Ex $y'' + xy = 0$: 2^{nd} order, linear, homogeneous, w/ variable coefficients

Ex $x^2 y'' - 2xy' + e^x y = 2x - 1$: 2nd order, linear, nonhomog. w/ variable coefficients

Ex $2y''' - 3y' + 7y = \ln(x^2 - 1)$

3rd order, linear

nonhomog., const coeff.

y, y', y'' : linear

$y^2, \sin y, e^y$: nonlinear

$ax + b$

$y^2 \cdot y''$: nonlinear

Ex $y''' + 2y'' - yy' + 7 = 0$

3rd order, nonlinear

We consider 2nd order linear homogeneous DE:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Ex (a) $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

(i) x^2 is a solution

$x^2 \cdot 2 - 2x \cdot 2x + 2 \cdot x^2 = 0$ ✓