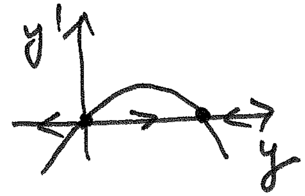
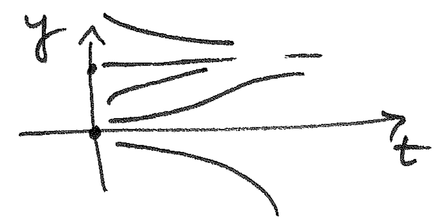
University of Idaho

Lecture 16

Review #1

- Separation of variables
- Method of integrating factor
- Mixture problem: set up an ODE and solve it
- Velocity problem
- Stability: critical points, phase diagram, solution curves





- Population models
- Numerical methods: Euler, modified Euler
  know the error order for Euler $O(h)$, modified Euler $O(h^2)$
  Runge-Kutta 4th order $O(h^4)$

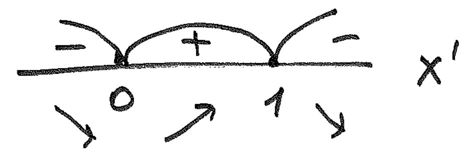
- $\frac{1}{2}$ page of your notes
- review formula sheet e.g. $e^{a+b} = e^a \cdot e^b$
- no calculator (you won't need one)

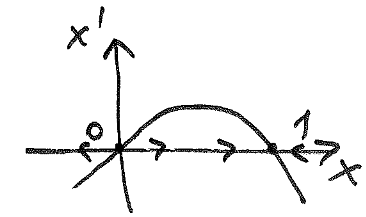
- review partial fraction decomposition

#1 S2.1

$$\frac{dx}{dt} = x - x^2, \qquad x(0) = 2$$

$$x - x^2 = 0$$
$$x(1-x) = 0 \; : \text{ two crit. pts } \; x=0 \;\&\; x=1$$
$$-x(x-1) = 0$$





$x=0$: stable equil. sol$^{\text{n}}$

$x=1$: unstable equil. sol$^{\text{n}}$

$$\frac{dx}{x-x^2} = dt, \qquad x \neq 0, \; x \neq 1$$

$(1-x \qquad (x$

$$\frac{1}{x-x^2} = \frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} \; : \text{ partial fraction decomposition}$$

$$1 = A(1-x) + Bx$$

$$x^1: \; 0 = -A + B \Rightarrow B = A$$

$$x^0: \; 1 = A \Rightarrow A = 1 \Rightarrow A = B = 1$$

$$\Rightarrow \frac{1}{x - x^2} = \frac{1}{x} + \frac{1}{1-x}$$

$$\int \frac{dx}{1-x} = \left| \begin{matrix} u = 1-x \\ du = -dx \end{matrix} \right|$$

$$\int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx = dt$$

$$\ln a - \ln b = \ln \frac{a}{b}$$

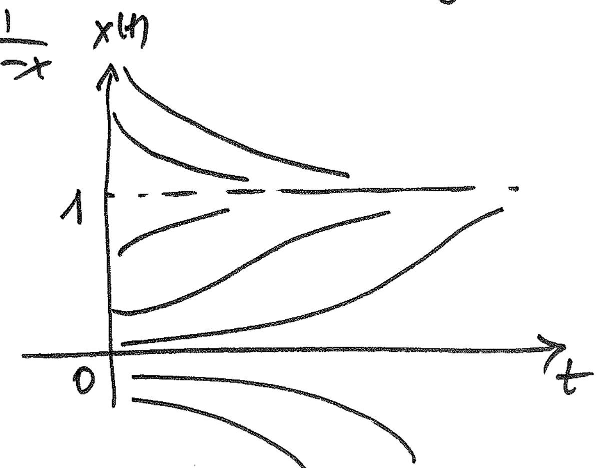
$$\ln|x| - \ln|1-x| = t + \tilde{C}$$

$$e^{\ln t} = t$$

$$\ln\left|\frac{x}{1-x}\right| = t + \tilde{C} \qquad \Big|\, \exp$$

$$\frac{x}{x-1} = Ce^{t} \qquad \Big| \cdot (x-1)$$

$$\text{at } t=0 \quad x(0) = 2 \Rightarrow \frac{2}{2-1} = C \Rightarrow \boxed{C=2}$$



$$\ln\left|\frac{x}{1-x}\right| = t + \tilde{C}$$

$$\left|\frac{x}{1-x}\right| = \underbrace{e^{t}}_{>0} \cdot \underbrace{e^{\tilde{C}}}_{>0}$$

$$x = Ce^{t}(x-1)$$

$$x\left[1 - Ce^{t}\right] = -Ce^{t}$$

$$x(t) = -\frac{Ce^{t}}{1-Ce^{t}} = -\frac{2e^{t}}{1-2e^{t}} = -\frac{2}{e^{-t}-2} = \boxed{\frac{2}{2-e^{-t}}}$$

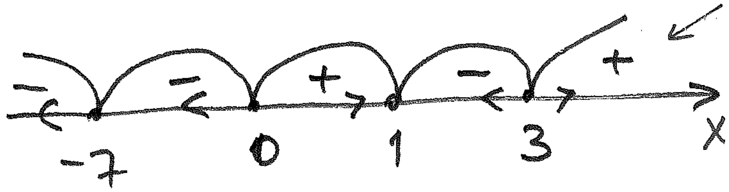
Ex $\dfrac{dx}{dt} = 5(x-3)(x+7)^2 x^3 (x-1)$

crit. points: $x_1 = -7$ $\quad m_1 = 2$: multiplicity

$x_2 = 0 \quad m_2 = 3$

$x_3 = 1 \quad m_3 = 1$

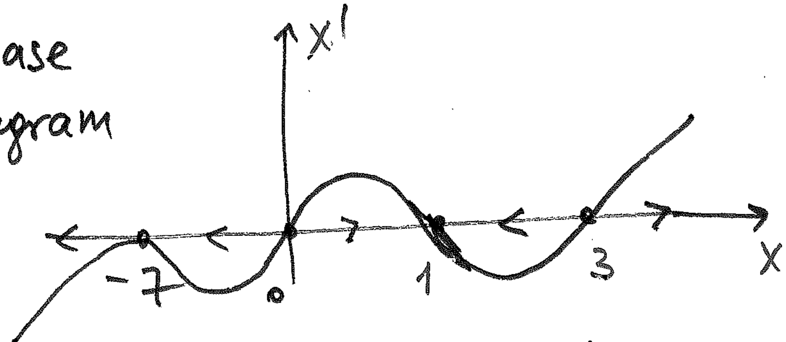
$x_4 = 3 \quad m_4 = 1$

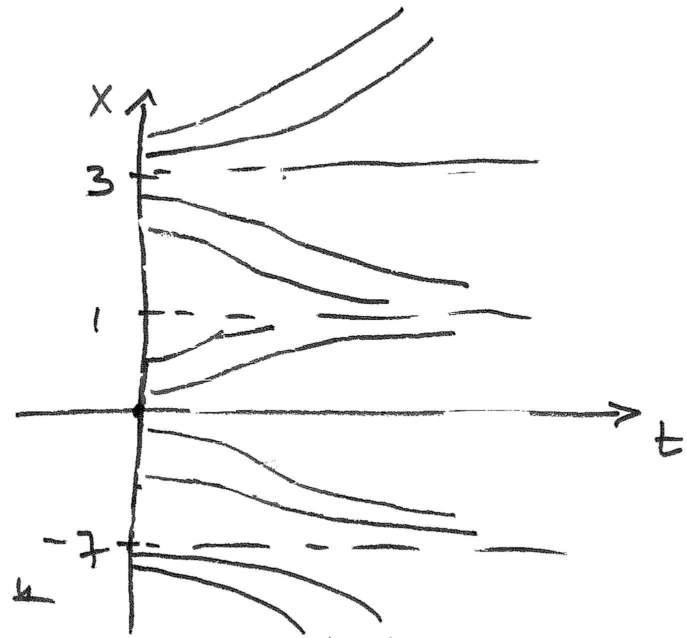


since $5>0$

x'

phase diagram





solution curves

$x=-7$: semi-stable / unstable equil. sol^{n}

$x=0$: unstable equil. sol^{n}

$x=1$: stable

$x=3$: unstable

#19 § 2.3

Find $v(10)$ & $\lim_{t\to\infty} v(t)$: limiting velocity

$v(0) = v_0 = 0$

$$\int \frac{du}{a^2-v^2} = \frac{1}{a}\underbrace{\operatorname{arctanh}}_{\tanh^{-1}}\frac{v}{a} + C = \frac{1}{a}\tanh^{-1}\frac{v}{a} + C$$

$$\frac{dv}{dt} = 4 - \frac{v^2}{400}$$

$$\frac{dv}{4-\frac{v^2}{400}} = dt$$

$$\frac{400\,dv}{40^2-v^2} = dt$$

$a = 40$

$$\frac{400}{40}\tanh^{-1}\frac{v}{40} = t + C$$

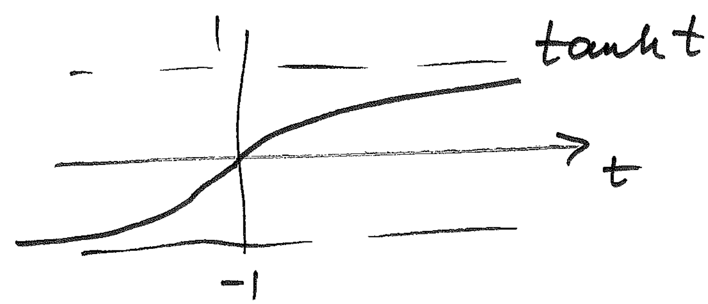
$$10\tanh^{-1}\frac{v}{40} = t + C$$

$v(0) = 0 \Rightarrow C = 0$

$$\Rightarrow \tanh^{-1}\frac{v}{40} = \frac{t}{10}$$

$$\frac{v}{40} = \tanh\frac{t}{10}$$

$$v(t) = 40\tanh\frac{t}{10}$$

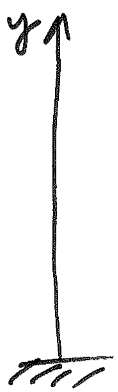


$$v(10) = 40\tanh 1 = 40\cdot\frac{e^1-e^{-1}}{e^1+e^{-1}} = 40\cdot(0{,}7616) = 30.4638 \ (ft/s)$$

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 40\tanh\frac{t}{10} = 40\lim_{t\to\infty}\tanh\frac{t}{10} = 40 \ ft/s$$

#11 S2.3

$1\ mi = 5280\ ft$



$y(0) = 1200\ ft = 0.227\ mi$

$v(8) = -100\ mi/h = -146.6667\ ft/s$

$y(8) = 0\ ft$

Find drag coefficient ρ assuming terminal velocity $v_\tau = -100\ mi/h$

$$\frac{dv}{dt} = -\rho v - g \qquad v_\tau = \lim_{t\to\infty} v(t) = -\frac{g}{\rho} \qquad \rho = \frac{k}{m} > 0$$

$$v_\tau = -100\ \frac{mi}{h} = -100 \cdot \frac{5280}{3600}\ ft/s = -146.6667\ ft/s$$

$$\rho = -\frac{g}{v_\tau} = \frac{32\ ft/s^2}{146.7\ ft/s} = 0.2181\ (1/s)$$

$$\frac{dv}{dt} = -\rho v - g \Rightarrow v(t) = \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} - \frac{g}{\rho}$$

$$y(t) = y_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t})$$

$$y(t) = 1200 - (146.7)t + 4.5844(0 + 146.7)(1 - e^{-0.2181 \cdot t})$$

$y = 0$ at the ground $\Rightarrow$ $1200 - (146.7)t + (4.5844)(146.7)(1 - e^{-0.2181 \cdot t}) = 0$

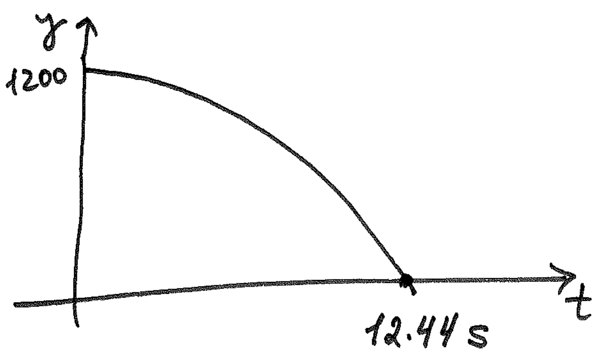
find $t = 12.44\ s$

Matlab code

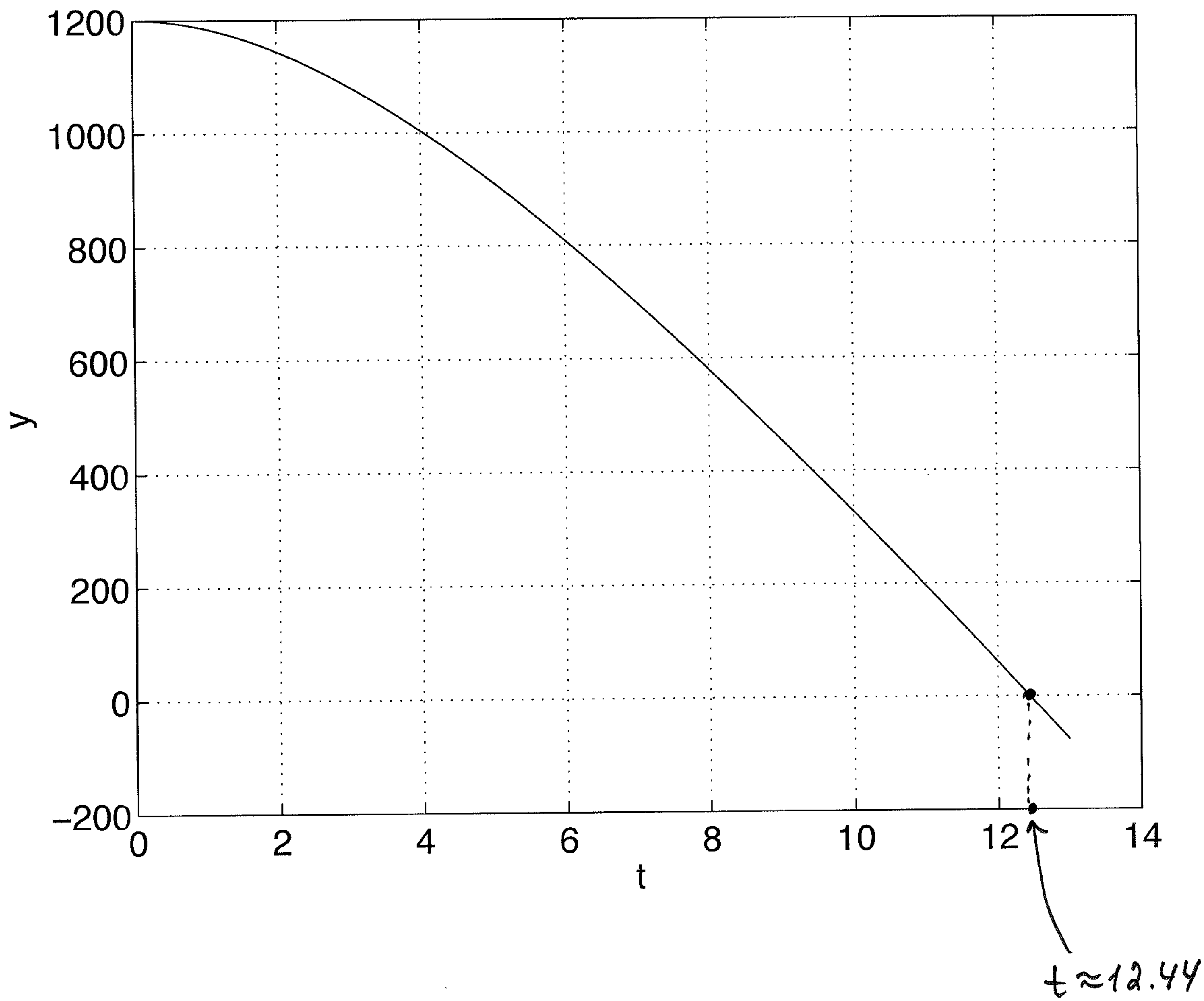
```
yo = 1200;
vo = 0;
vtau = -100*5280/3600;
g = 32.174;
rho = -g/vtau;      t=0:1e-2:13;
y=yo+vtau*t+1/rho*(vo-vtau)*
       *(1-exp(-rho*t))
plot(t,y)  xlim([12.2 12.6])
```

to zoom in

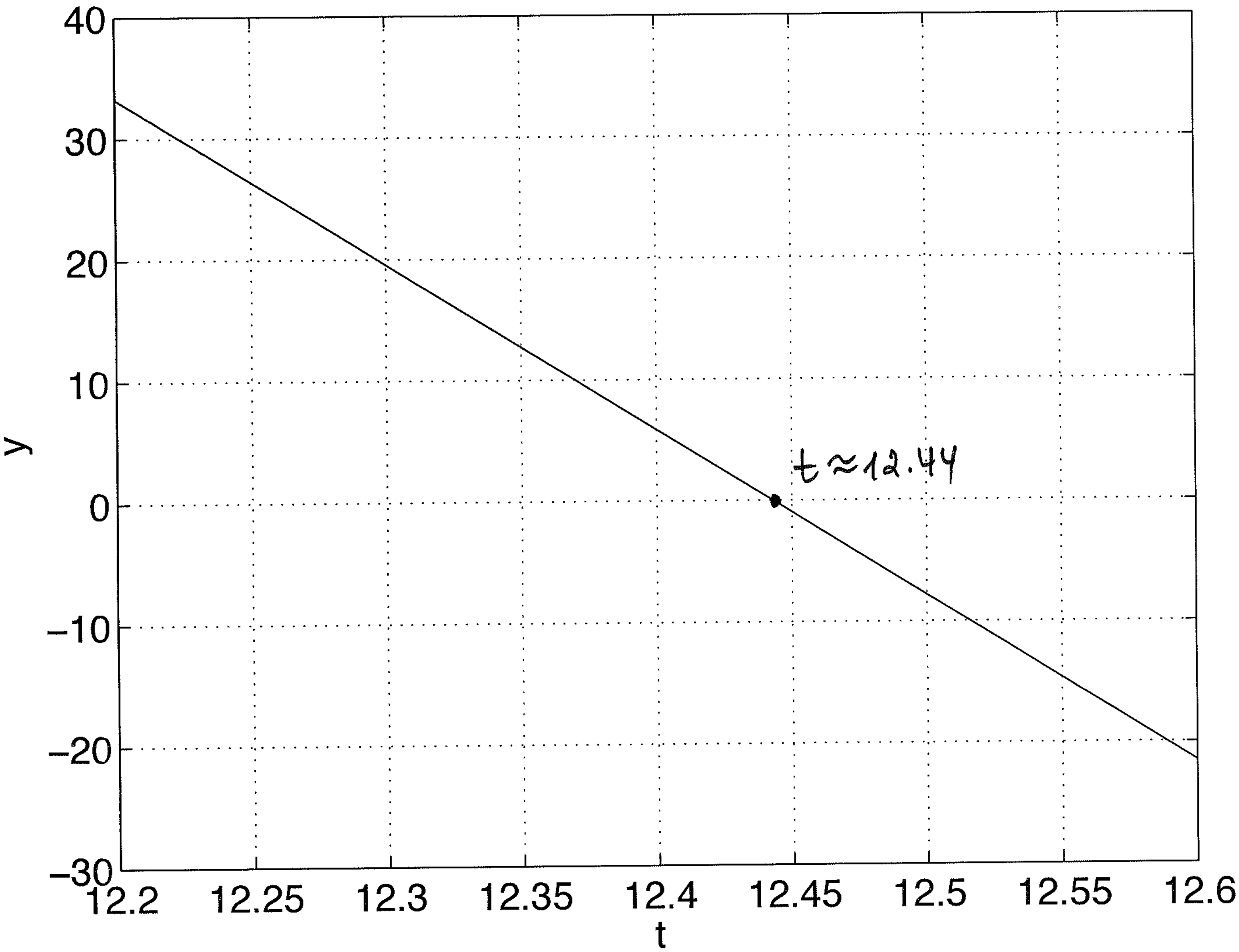
Graph $y(t)$.



Problem 11, Section 2.3. Vertical displacement y(t)



Problem 11, Section 2.3. Vertical displacement y(t) (zoom-in)



```
clear all

y0=1200;
v0=0;
vtau=-100*5280/3600;

g=32.174;
rho=-g/vtau;

t=0:1e-2:13;

y=y0+vtau*t+1/rho*(v0-vtau)*(1-exp(-rho*t));

figure(1);clf(1)
set(gca,'FontSize',16); set(gca,'box','on')

plot(t,y)
grid on
xlabel('t')
ylabel('y')
title('Problem 11, Section 2.3. Vertical displacement y(t)')
```

#37 S1.5 A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

$V_{max} = 400$ gal

$V(0) = 100$ gal

$x(0) = 50$ lb

$c_{in} = 1$ lb/gal, $r_{in} = 5$ gal/s, $r_{out} = 3$ gal/s

$\frac{dV}{dt} = r_{in} - r_{out} \Rightarrow \frac{dV}{dt} = 5 - 3 = 2$

$V = 2t + C \qquad V(0) = 100 \Rightarrow C = 100$

$\boxed{V = 2t + 100}$

$\frac{dx}{dt} = c_{in} r_{in} - c_{out} r_{out}$

$c_{out} = \frac{x(t)}{V(t)} = c(t)$

$$\frac{dx}{dt} = 1 \cdot 5 - 3 \cdot \frac{x}{2t+100}$$

$$\frac{dx}{dt} + \underbrace{\frac{3}{2t+100}}_{P(t)} x = \underbrace{5}_{Q(t)}$$

$a \ln x = \ln x^a$

$$\rho(t) = e^{\int P(t)\,dt} = e^{\int \frac{3}{2t+100}\,dt} = e^{\frac{3}{2}\ln|2t+100|} = (2t+100)^{3/2}, \text{ since } t \geq 0$$

$$\rho x = \int \rho Q\,dt + C$$

$$(2t+100)^{3/2} \cdot x(t) = \int (2t+100)^{3/2} \cdot 5\,dt + C$$

$$(2t+100)^{3/2} \cdot x(t) = \frac{5}{\cancel{2}} \frac{1}{\frac{3}{2}\cancel{+1}} (2t+100)^{5/2} + C$$

$$(2t+100)^{3/2} x(t) = (2t+100)^{5/2} + C \quad \Big| \; (2t+100)^{-3/2}$$

$$x(0) = 50 \Rightarrow 100^{3/2} \cdot \underbrace{50}_{= \frac{100}{2}} = 100^{5/2} + C \Rightarrow \frac{1}{2} 100^{5/2} = 100^{5/2} + C \Rightarrow -\frac{1}{2} 100^{5/2} = C$$

$$x(t) = 2t + 100 - (2t+100)^{-3/2} \cdot \underbrace{\frac{1}{2} 100^{5/2}}_{=50 \cdot 100^{3/2}} = \boxed{2t + 100 - 50\left(\frac{100}{2t+100}\right)^{3/2} = x(t)}$$

Tank is full when $V(t) = V_{max} \Rightarrow 2t + 100 = 400 \Rightarrow t = 150$ s

$$x(150) = 2 \cdot 150 + 100 - 50\left(\frac{100}{2 \cdot 150 + 100}\right)^{3/2} = 400 - 50\left(\frac{100}{400}\right)^{3/2} = 400 - 50\left(\frac{1}{2^2}\right)^{3/2} = 400 - \frac{50}{8} =$$

$$= \boxed{393.75 \text{ (lb)}}$$