Lecture 17

Ch. 3 Linear Equations of Higher Order (Cont'd)

Ex (a) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

(i) We showed last time that x^2 is a solution

(ii) $C_1 x^2$ is a solution

$$x^2 \cdot 2C_1 - 2x \cdot 2C_1 x + 2 \cdot C_1 x^2 \stackrel{?}{=} 0 \qquad 0 = 0 \checkmark$$

(iii) x is a solution

$$x^2 \cdot 0 - 2x \cdot 1 + 2 \cdot x = 0 \quad \checkmark$$

(iv) $C_2 x$ is a solution

$$x^2 \cdot 0 - 2x \cdot C_2 + 2 \cdot C_2 x = 0 \quad \checkmark$$

(v) $C_1 x^2 + C_2 x$ is also a solution

$$x^2 \cdot 2C_1 - 2x\,(2C_1 x + C_2) + 2\,(C_1 x^2 + C_2 x) \stackrel{?}{=} 0 \qquad C_1 \cdot 0 + C_2 \cdot 0 = 0$$

Ex (6) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$: 2nd order DE, homogeneous w/ const coefficients

(i) e^{-x} is a solution

$e^{-x} + 3(-e^{-x}) + 2 \cdot e^{-x} = 0$ ✓

(ii) $C_1 e^{-x}$ is a solution

(iii) e^{-2x} is a solution

(iv) $C_2 e^{-2x}$ — " —

(v) $C_1 e^{-x} + C_2 e^{-2x}$ — " —

Thm Principle of Linear Superposition

Given

$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0$: 2nd order, linear homog. DE

If $y_1(x)$ and $y_2(x)$ are solutions of this DE, then their linear combination

$$C_1 y_1(x) + C_2 y_2(x)$$

is also a solution of this DE, where C_1 and C_2 are arbitrary constants.

In general, for n^{th} order linear homogeneous DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

if $y_1(x), y_2(x), \dots, y_n(x)$ are solutions of this DE, then their linear combination

$$C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$$

is also a solution of this DE, where $C_1, C_2, \dots, C_n$ are arbitrary constants.

Operator notation for linear DEs with constant coefficients

Denote by $D = \frac{d}{dx}$, then

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = DD = D^2$$

Then we can write a DE w/ constant coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

as

$$a_n D^n y + a_{n-1} D^{n-1} y + \dots + a_2 D^2 y + a_1 D y + \underset{\substack{\parallel \\ D^0 y}}{a_0 y} = 0$$

or

$$\underbrace{\left(a_n D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0\right)}_{P(D)} y = 0$$

$P(D)$ is a differential operator

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0$$ is

a polynomial in D of degree n

Note every linear homogeneous DE w/ constant coefficients can be written as $P(D)y = 0$ and vice versa, for every operator polynomial $P(D)$, there is a DE $P(D)y=0$: linear DE, homogeneous, w/ const coefficients.

Ex Write the following DEs using operator notation.

(a) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 13y = 0$

$(D^2 + 6D + 13)y = 0$

(b) $y^{IV} + y''' - 64y'' + 17y' - 6y = 0$

$(D^4 + D^3 - 64D^2 + 17D - 6)y = 0$

Thm

$$(D-a)(D-b)y = (D-b)(D-a)y = \left[D^2-(a+b)D+ab\right]y$$

i.e.

$$\left(\frac{d}{dx}-a\right)\left(\frac{d}{dx}-b\right)y = \left(\frac{d}{dx}-b\right)\left(\frac{d}{dx}-a\right)y = \left(\frac{d^2}{dx^2}-(a+b)\frac{d}{dx}+ab\right)y$$

$\square$

$$\left(\frac{d}{dx}-a\right)\left(\frac{d}{dx}-b\right)y = \left(\frac{d}{dx}-a\right)\left(\frac{dy}{dx}-by\right) = \frac{d^2y}{dx^2} - b\frac{dy}{dx} - a\left(\frac{dy}{dx}-by\right)$$

$$= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + ab\cdot y = \left[D^2-(a+b)D+ab\right]y \quad \blacksquare$$

Note This shows a property of the operator $P(D)$ which it shares with polynomials, i.e. $P(D)$ can be written in a form as if "polynomial" $P(D)$ were factored out. It is true for polynomials $P(D)$ of any degree.

Ex $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)y = 0$

$(D-1)(D-2)(D-3)y = 0$

or $(D-2)(D-3)(D-1)y = 0$

or $(D-3)(D-1)(D-2)y = 0$

Ex evaluate $D^2 + 2D + 1$ operating on e^x, $\sin x + 3$, e^{-x}, xe^{-x}.

$(D^2+2D+1)e^x = e^x + 2e^x + 1\cdot e^x = 4e^x$

$(D^2+2D+1)(\sin x + 3) = -\cancel{\sin x} + 2\cdot\cos x + (\cancel{\sin x} + 3) = 2\cos x + 3$

$(D^2+2D+1)e^{-x} = e^{-x} + 2(-e^{-x}) + 1\cdot e^{-x} = 0 \longleftarrow$

$(D^2+2D+1)(xe^{-x}) = (-\cancel{e^{-x}} - \cancel{e^{-x}} + \cancel{xe^{-x}}) + 2(\cancel{e^{-x}} - \cancel{xe^{-x}}) + 1\cdot(\cancel{xe^{-x}}) = 0 \longleftarrow$

In this example, we saw that operator D^2+2D+1 when operating on e^{-x} and xe^{-x} produced 0. But this is precisely what we need in order to solve the DE

$$(D^2+2D+1)y=0 \qquad (*)$$

Hence, functions e^{-x} and xe^{-x} are solutions of $(D^2+2D+1)y=0$. This means that to solve DE $(D^2+2D+1)y=0$ we seek functions that the operator D^2+2D+1 ANNIHILATES.

<u>Note</u> Since e^{-x} and xe^{-x} are solutions of $(D^2+2D+1)y=0$ or $y''+2y'+y=0$, their linear combination

$$C_1e^{-x}+C_2xe^{-x}$$

is also a solution of DE $(*)$.

Ex Solve

$$y''' - 6y'' + 11y' - 6y = 0$$

$$(D^3 - 6D^2 + 11D - 6)y = 0$$

$$(D-1)(D-2)(D-3)y = 0$$

Since we can permute the order of $(D-1)$, $(D-2)$, $(D-3)$, we need to find a function that operator $D-a$ will annihilate and then we will have three solutions.