Lecture 18

OPERATOR IDENTITY I $\boxed{(D-a)e^{ax}=0}$

Proof

$$(D-a)\,e^{ax} = ae^{ax} - a\cdot e^{ax} \equiv 0$$

$$\therefore\quad (D-1)(D-2)\underbrace{(D-3)e^{3x}}_{=0} = (D-1)(D-2)\,0 = 0$$

$\Rightarrow e^{3x}$ is a solution

$$(D-2)(D-3)\underbrace{(D-1)e^{x}}_{=0} = (D-2)(D-3)\,0 = 0$$

$\Rightarrow e^{x}$ is a solution

$$(D-3)(D-1)\underbrace{(D-2)e^{2x}}_{=0} = (D-3)(D-1)\,0 = 0$$

$\Rightarrow e^{2x}$ is a solution

The general solution of $y''' - 6y'' + 11y' - 6y = 0$ is

$$y(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$$

where C_1, C_2, C_3 are arbitrary constants and functions e^x, e^{2x}, e^{3x} are linearly independent (more - later).

Ex Solve $y'' - 4y' - 5y = 0$

$(D^2 - 4D - 5)y = 0$

$-1, 5$: roots

$(D+1)(D-5)y = 0$

$\boxed{(D-a)e^{ax} = 0}$

$ax^2 + bx + c = 0$

x_1, x_2: roots

$ax^2 + bx + c = a(x - x_1)(x - x_2)$

Vietta's Thm:

$x_1 \cdot x_2 = \frac{c}{a}$

$x_1 + x_2 = -\frac{b}{a}$

Solutions: e^{-x} and e^{5x}

General solution: $\boxed{y(x) = C_1 e^{-x} + C_2 e^{5x}}$

Alternative approach

Assume that solution has the form: $\boxed{y = e^{rx}}$

$y' = re^{rx}, \quad y'' = r^2 e^{rx}$

Substitute $y = e^{rx}$ into DE $y'' - 4y' - 5y = 0$

$$r^2 e^{rx} - 4r e^{rx} - 5e^{rx} = 0$$

$$\underbrace{e^{rx}}_{\neq 0} (r^2 - 4r - 5) = 0$$

$\therefore$ $\boxed{r^2 - 4r - 5 = 0}$ <u>characteristic equation</u>

compare w/ $(D^2 - 4D - 5)y = 0$
$-1, 5$

Roots are $r_1 = -1$, $r_2 = 5$

$\Rightarrow$ solutions are $e^{r_1 x} = e^{-x}$, $e^{r_2 x} = e^{5x}$: same as above

<u>Ex</u> Solve $y'' - 4y' - 5y = 0$ subject to ICs $y(0) = 5$, $y'(0) = 7$

The general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{5x}$$

$$y'(x) = -C_1 e^{-x} + 5C_2 e^{5x}$$

$y(0) = 5 \Rightarrow 5 = C_1 e^0 + C_2 e^0 \Rightarrow C_1 + C_2 = 5$

$y'(0) = 7 \Rightarrow 7 = -C_1 + 5C_2 \Rightarrow -C_1 + 5C_2 = 7$

system of 2 eq^ns + for 2 unknowns C_1, C_2

Y

$6C_2 = 12 \Rightarrow C_2 = 2$

$C_1 = 5 - C_2 = 5 - 2 = 3$

$$\Rightarrow \boxed{y(x) = 3e^{-x} + 2e^{5x}}$$

Repeated Real Roots

Ex Consider

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$(D^2 + 6D + 9)y = 0$$

$$(D+3)^2 y = 0$$

$-3, -3$

$x - x_0$

Solution: $y(x) = C_1 e^{-3x} + C_2 e^{-3x} = (\underbrace{C_1 + C_2}_{=C})e^{-3x} = Ce^{-3x}$

$\therefore$ We need another "different" (linearly independent) solution.

OPERATOR IDENTITY II

$$\boxed{(D-a)^n x^k e^{ax} = 0, \quad k = 0, 1, 2, \dots, n-1}$$

i.e. operator $(D-a)^n$ annihilates functions

$$e^{ax},\ xe^{ax},\ x^2e^{ax},\ \dots,\ x^{n-1}e^{ax}$$

Ex $(D-5)^4$ annihilates $e^{5x}, xe^{5x}, x^2e^{5x}, x^3e^{5x}$

5,5,5,5

$(D+1)^6$ annihilates $e^{-x}, xe^{-x}, x^2e^{-x}, x^3e^{-x}, x^4e^{-x}, x^5e^{-x}$

$\underbrace{-1,-1,-1,-1,-1,-1}_{6 \text{ times}}$

$(D^2-4^2)^3 = (D-4)^3(D+4)^3$ annihilates $e^{4x}, xe^{4x}, x^2e^{4x},$
$e^{-4x}, xe^{-4x}, x^2e^{-4x}$

University of Idaho

Returning to

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$(D^2 + 6D + 9)y = 0$$

$$(D+3)^2 y = 0$$

$-3, -3$

Solutions: e^{-3x} and xe^{-3x}

General solution: $y(x) = C_1 e^{-3x} + C_2 x e^{-3x}$

Ex $D^3(D^2-1)(D^2+3D+2)y = 0$

$(D-0)^3(D+1)(D-1)(D+1)(D+2)y = 0$

$(D-0)^3(D-1)(D+1)^2(D+2)y = 0$

$0, 0, 0, 1, -1, -1, -2$

$(D-a)e^{ax} = 0$

$a = 0$

$e^{0\cdot x}, xe^{0\cdot x}, x^2 e^{0\cdot x}$

or

$1, x, x^2$

$y(x) = C_1 e^{0\cdot x} + C_2 x e^{0\cdot x} + C_3 x^2 e^{0\cdot x} + C_4 e^{x} + C_5 e^{-x} + C_6 x e^{-x} + C_7 e^{-2x}$

or

$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^{x} + C_5 e^{-x} + C_6 x e^{-x} + C_7 e^{-2x}$

Ex $y''' - 4y'' + 4y' = 0$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = -4$

$(D^3 - 4D^2 + 4D)y = 0$

$D(D^2 - 4D + 4)y = 0$

$D(D-2)^2 y = 0$

$0, 2, 2$

$y(x) = C_1 e^{0\cdot x} + C_2 e^{2x} + C_3 x e^{2x}$

$\boxed{y(x) = C_1 + C_2 e^{2x} + C_3 x e^{2x}}$

$y' = 2C_2 e^{2x} + C_3 e^{2x} + C_3 x \cdot 2 e^{2x}$

$\boxed{y'_{(x)} = (2C_2 + C_3) e^{2x} + 2C_3 x e^{2x}}$

$\boxed{y''(x) = 4(C_2 + C_3) e^{2x} + 4C_3 x e^{2x}}$

at $x=0$:

$$\left.\begin{array}{lll} y(0)=3 \Rightarrow & 3 = C_1 + C_2 e^0 + \cancel{C_3 \cdot 0 e^0} & \Rightarrow C_1 + C_2 = 3 \\ y'(0)=1 \Rightarrow & 1 = (2C_2 + C_3)e^0 + \cancel{2C_3 \cdot 0 e^0} & \Rightarrow 2C_2 + C_3 = 1 \\ y''(0)=-4 \Rightarrow & -4 = 4(C_2+C_3)\underbrace{e^0}_{1} + \cancel{4C_3 \cdot 0 e^0} & \Rightarrow C_2 + C_3 = -1 \end{array}\right\}$$

Solve for C_1, C_2, C_3:

$C_1 = 1,\quad C_2 = 2,\quad C_3 = -3$

$\therefore\ \boxed{y(x) = 1 + 2e^{2x} - 3xe^{2x}}$

Imaginary Roots

Ex $y'' + 4y = 0$

$(\underset{\pm 2i}{D^2+4})y = 0$

$D^2 + 4 = 0$

$D^2 = -4 \qquad -1 = i^2$

$D = \pm 2i$

We could write : $y(x) = C_1 e^{2ix} + C_2 e^{-2ix}$

Q How to evaluate e^{2ix} ?

To do this, we recall the Taylor expansion of e^x around $x=0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$: converges for all x

Define

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \quad =$$

$i^2 = -1$, $\quad i^3 = i^2 \cdot i = -i$, $\quad i^4 = i^2 \cdot i^2 = 1$, $\quad i^5 = i^4 \cdot i = i$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots$$

$z = x + iy$: complex number

$x = \operatorname{Re} z$: real part of z

$y = \operatorname{Im} z$: imaginary part of z