Lecture 19

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

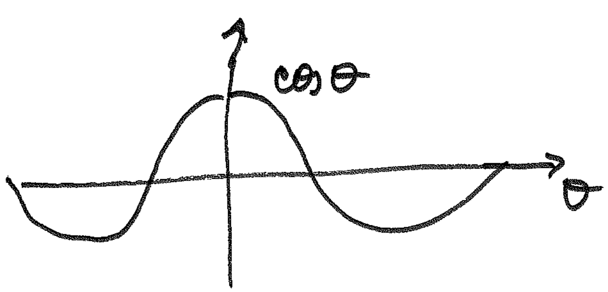
But we know (or should!) these series. These are Taylor series expansions of $\cos\theta$ and $\sin\theta$ around $\theta = 0$.

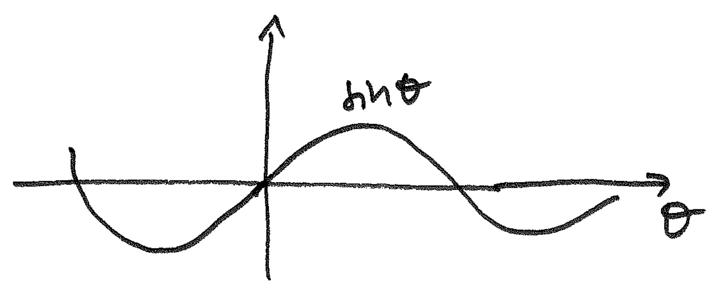
$$\Rightarrow \boxed{\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}}$$

Euler's formulas

$\cos(-\theta) = \cos\theta$
even function

$\sin(-\theta) = -\sin\theta$
odd function





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Back to example

$z = x + iy$

$\bar{z} = x - iy$: complex conjugate of $\bar{z}$

$y'' + 4y = 0$

$(D^2 + 4)y = 0$

$\pm 2i$

$$y(x) = C_1 e^{2ix} + C_2 e^{-2ix} \underset{\text{Euler's formulas}}{=} C_1(\cos 2x + i\sin 2x) + C_2(\cos 2x - i\sin 2x) =$$

$$= \underbrace{(C_1 + C_2)}_{=K_1} \cos 2x + \underbrace{i(C_1 - C_2)}_{K_2} \sin 2x = K_1 \cos 2x + K_2 \sin 2x$$

But this solution is complex-valued. We can verify directly that $\cos 2x$ and $\sin 2x$ are also solutions of $y'' + 4y = 0$.

$y = \cos 2x \qquad y' = -2\sin 2x, \quad y'' = -4\cos 2x$

$-4\cos 2x + 4 \cdot \cos 2x \equiv 0 \Rightarrow y = \cos 2x$ is a solution of $y'' + 4y = 0$

Similarly, for $y = \sin 2x$

General solution: $\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x}$

OPERATOR IDENTITY III

$$\underset{\pm ib}{(D^2+b^2)} \begin{matrix} \cos bx \\ \sin bx \end{matrix} = 0$$

$$(D^2+b^2)\cos bx =$$
$$= -b^2 \cos bx + b^2 \cdot \cos bx = 0$$

OPERATOR IDENTITY IV

$$(D^2+b^2)^n \begin{matrix} x^k \cos bx \\ x^k \sin bx \end{matrix} = 0 \qquad k = 0, 1, \ldots, n-1$$

$$\underbrace{\pm ib, \pm ib, \ldots, \pm ib}_{n \text{ times}}$$

Solve

$$y'' + 9y = 0$$

$$(D^2 + 9)y = 0$$

$$\pm 3i$$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x$$

$$2y'' + 6y = 0$$

$$(2D^2 + 6)y = 0$$

$$2(D^2 + 3)y = 0$$

$$\pm\sqrt{3}i$$

$$y(x) = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x$$

$$2y'' - 6y = 0$$

$$(2D^2 - 6)y = 0$$

$$2(D^2 - 3)y = 0$$

$$\pm\sqrt{3}$$

$$y(x) = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x}$$

Ex Solve $y^{IV} + 8y'' + 16y = 0$

$$(D^4 + 8D^2 + 16)y = 0$$

$$(D^2 + 4)^2 y = 0$$

$$\pm 2i, \pm 2i$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 x \cos 2x + C_4 x \sin 2x$$

Ex Solve

$(D-1)^2 D^4 (D+1)^3 (D^2+2)(D^2+1)^3 (D^2-4)\, y = 0$ 19th order

$1, 1, 0, 0, 0, 0, -1, -1, -1, \pm i\sqrt{2}, \pm i, \pm i, \pm i, 2, -2$

General solution $e^{0 \cdot x} = 1$

$$y(x) = C_1 e^x + C_2 x e^x + C_3 + C_4 x + C_5 x^2 + C_6 x^3 +$$
$$+ C_7 e^{-x} + C_8 x e^{-x} + C_9 x^2 e^{-x} + C_{10} \cos\sqrt{2}x + C_{11} \sin\sqrt{2}x +$$
$$+ C_{12} \cos x + C_{13} \sin x + C_{14} x \cos x + C_{15} x \sin x + C_{16} x^2 \cos x + C_{17} x^2 \sin x +$$
$$+ C_{18} e^{2x} + C_{19} e^{-2x}$$

19 arbitrary constants