Math 310 : ODEs

Lecture 1

1.1. DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

Natural phenomena involve some changes (in time or/and space) and they can be modeled by differential equations.

Consider $x = f(t)$: function of t

t: independent variable

x: dependent variable

$\frac{df}{dt} = f'(t)$: rate of change of f wrt t

Def A <u>differential equation</u> (DE) is an equation that involves an unknown function and one or more of its derivatives.

Ex $\frac{dx}{dt} = x + \sin t$

$\uparrow$

1^{st} order derivative is the highest order derivative

$\Rightarrow$ DE is of 1^{st} order.

$x = x(t)$

t: independent variable

x: dependent — " —

Aside:

$y = ax + b$: linear function

Ex $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$

highest derivative is of order 2

$\Rightarrow$ 2nd order DE.

$y = y(x)$

x: indep. var.

y: dep. var.

Algebra problem: solve algebraic equation $x^2 = 4$.

Solution $x = \pm 2$. When we substitute $x = \pm 2$ into eqn $x^2 = 4$, we get $(\pm 2)^2 = 4$, i.e. $4 = 4$ – an identity.

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Def To solve a DE, we need to find a function $y = y(x)$ that when substituted into the DE, produces an identity for all $x \in I$: some interval. (Plus it may have to satisfy other conditions — more later (initial conditions))

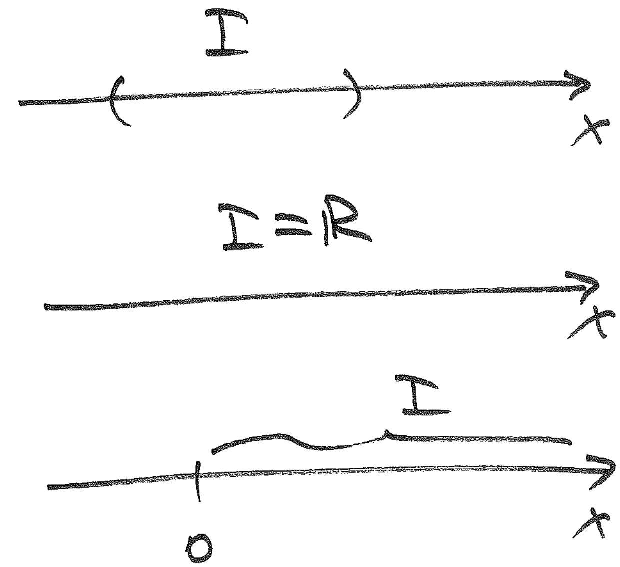
belongs

Ex

$y(x) = Ce^{x^2}$, C: const

$$\frac{dy}{dx} = \underbrace{Ce^{x^2}}_{y} \cdot 2x \Rightarrow \frac{dy}{dx} = 2xy$$

1st order DE for $y(x)$



$\Rightarrow y(x) = Ce^{x^2}$ satisfies DE $\frac{dy}{dx} = 2xy$, for all $x \in \mathbb{R}$.

We can check directly if $y = Ce^{x^2}$ is a solution of $\frac{dy}{dx} = 2xy$.

$$y' = Ce^{x^2} \cdot 2x = \frac{dy}{dx}$$

$$\frac{dy}{dx} = 2xy$$

$$Ce^{x^2} \cdot 2x = 2x \cdot Ce^{x^2} \checkmark$$

identity for all x

$\therefore$ $y = Ce^{x^2}$ is a solution of

$$\frac{dy}{dx} = 2xy.$$

Note. Function $y(x) = Ce^{x^2}$ defines infinitely many solutions of $\frac{dy}{dx} = 2xy$, that depend on one parameter C. We say that we have a one-parameter family of solutions.

Note 1st order DE has one-parameter family of solutions, i.e. solutions depend on one arbitrary constant.

Ex <u>Newton's Law of Cooling</u>

A: temperature of surrounding medium

T(t): temperature of an object

$\frac{dT}{dt}$: rate of change of T wrt t

Newton's Law of Cooling: rate of change of T wrt t is proportional to $T-A$, i.e. the difference between $T(t)$ and the temperature A of the surrounding medium.

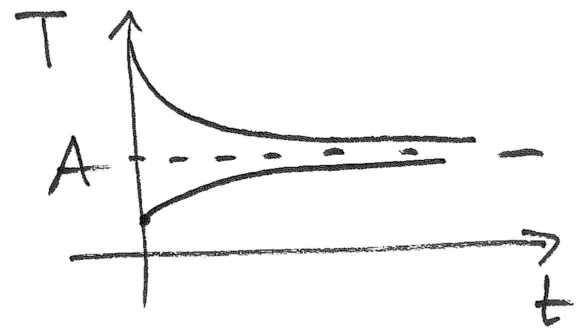
$$\frac{dT}{dt} = \underbrace{-k}_{\text{const}}(T-A) \qquad k>0$$

if $T>A \Rightarrow T-A>0$, $\underbrace{-k}_{<0}\underbrace{(T-A)}_{>0} < 0$

$\Rightarrow \frac{dT}{dt} < 0 \Rightarrow T \searrow$ (temperature decreases)

if $T<A \Rightarrow T-A<0$, $-k(T-A)>0 \Rightarrow \frac{dT}{dt}>0$

$\Rightarrow T \nearrow$ as $t \nearrow$, i.e. T increases with time.



Ex <u>Population Model</u>

$P(t)$: population size

Rate of change of population is proportional to the population size itself.

$$\frac{dP}{dt} = kP \qquad k: \text{const} \qquad (1)$$

Note that $P(t) = Ce^{kt}$ is a solution of eqn (1).

C is an arbitrary const.

We can verify that $P(t) = Ce^{kt}$ is a solution of (1).

Indeed,

$$\frac{dP}{dt} = Ce^{kt} \cdot k \qquad \qquad \frac{dP}{dt} = kP$$

$$Ce^{kt} \cdot k = k \cdot Ce^{kt} \quad \checkmark$$

identity

$\therefore \quad P(t) = Ce^{kt}$ is a solution of $\frac{dP}{dt} = kP$.

<u>Ex</u> $P(t)$: population that satisfies $\frac{dP}{dt} = kP$

Solution is $P(t) = Ce^{kt}$.

at $t=0$ $\quad P(0) = 1000$: <u>initial condition</u>

at $t=1$ $\quad P(1)$ doubled, i.e. $P(1) = 2000$

Find population size $P(t)$ at any time.

$P(t) = Ce^{kt}$

at $t=0$ $\quad P(0) = 1000$

$$\underset{\substack{\| \\ 1000}}{P(0)} = C\underbrace{e^{k \cdot 0}}_{1} = C \Rightarrow \boxed{C = 1000}$$

$$\boxed{P(t) = 1000e^{kt}}$$

at $t=1$ $\quad P(1) = 2000$

$$\underset{\substack{\| \\ 2000}}{P(1)} = 1000e^{k \cdot 1} \Rightarrow 1000e^{k} = 2000$$

$$e^{k} = 2 \quad | \ln$$

$$\boxed{k = \ln 2}$$

$$\ln e^{k} = k$$

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$$\therefore \quad \boxed{P(t) = 1000\, e^{\ln 2 \cdot t}}$$