Lecture 1

1.1 DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

Natural phenomena involve some changes (in time or/and in space) and they can be modeled by differential equations.

$x = f(t)$

t: independent variable

x: dependent variable

$f'(t) = \frac{df}{dt}$: rate of change of $f(t)$ wrt t

Def A *differential equation* (DE) is an equation that relates an uknown function and one or more of its derivatives.

Ex $\frac{dx}{dt} = x + \sin t$

1st order derivative

t: independent variable

x: dependent variable

$x = x(t)$

This is a 1st order DE

Ex $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$

highest derivative is of order 2

$y = y(x)$

x: independent variable

y: dependent variable

$\Rightarrow$ 2nd order DE.

<u>algebra problem</u>: solve algebraic equation $x^2=4$. Solution $x=\pm 2$.

When we substitute solution $x=\pm 2$ into eq^n^ $x^2=4$, we get $(\pm 2)^2 = 4$, i.e. $4 \equiv 4$: an identity.

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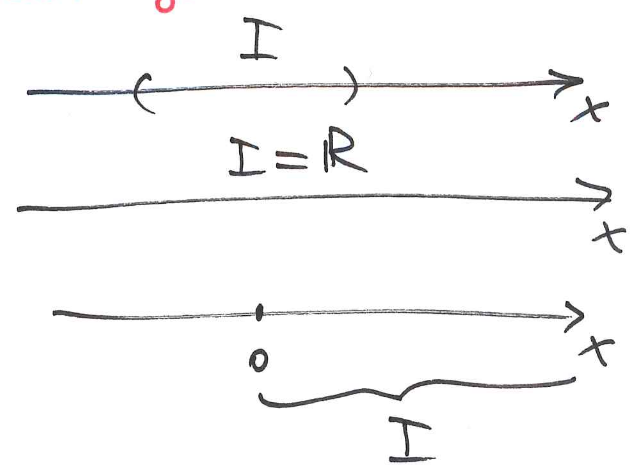
Def To solve a DE, we need to find a function $y=y(x)$ that when substituted into the DE, produces an identity for all $x \in I$ – some interval. (plus it may have to satisfy other conditions – initial conditions – more later).

$x \in I$ ↑ belongs to



Ex $y(x) = Ce^{x^2}$, C is a constant

$$y' = \frac{dy}{dx} = \underbrace{Ce^{x^2}}_{y} \cdot 2x = y \cdot 2x$$

$\Rightarrow y' = 2xy$: this is a DE of order 1

$\Rightarrow y(x) = Ce^{x^2}$ satisfies DE $y' = 2xy$

or $y = Ce^{x^2}$ is a solution of DE $y' = 2xy$, for all x.

We can also check that $y = Ce^{x^2}$ satisfies $y' = 2xy$.

To do this, we substitute $y = Ce^{x^2}$ into $y' = 2xy$ and check if we get an identity.

$y' = 2xy \qquad y = Ce^{x^2}, \qquad y' = Ce^{x^2} \cdot 2x$

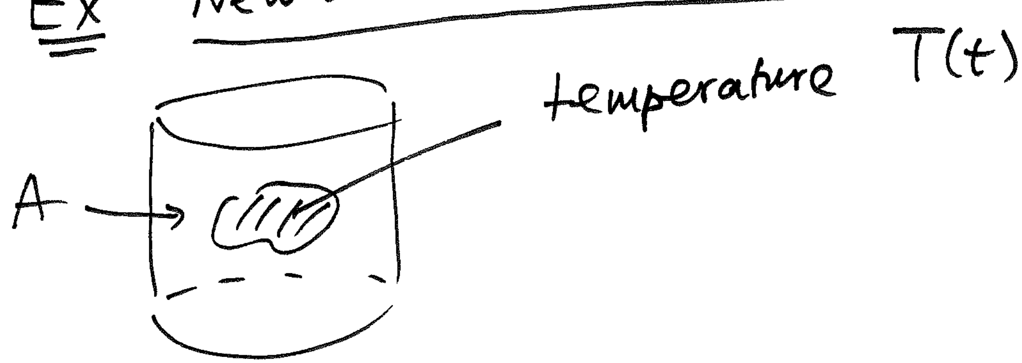
$Ce^{x^2} \cdot 2x = 2x \cdot Ce^{x^2}$ true for all $x \in \mathbb{R}$

$\Rightarrow$ $\boxed{y = Ce^{x^2}}$ is a solution of $y' = 2xy$ for all $x \in \mathbb{R}$.

$y(x) = Ce^{x^2}$ defines infinitely many solutions of $y' = 2xy$, that depend on parameter C. We say that $y(x) = Ce^{x^2}$ is a one-parameter family of solutions.

<u>Note</u> 1st order DE has one-parameter family of solutions, ie. solutions depend on one arbitrary constant.

<u>Ex</u> <u>Newton's Law of Cooling</u>



$T(t)$: temperature of object

A: temperature of surrounding medium

$\frac{dT}{dt}$: rate of change of T wrt t

<u>Newton's Law of Cooling</u> : rate of change of T wrt t is proportional to $T-A$, ie. difference between $T(t)$ and temperature A of surrounding medium.

$$\frac{dT}{dt} = -\underbrace{k}_{\text{const}}(T-A), \quad k>0$$

proportional

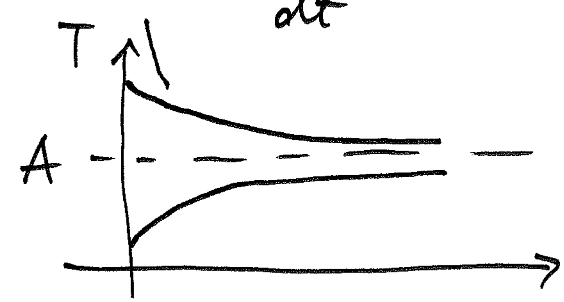
if $T>A \Rightarrow T-A>0 \Rightarrow -k(T-A)<0 \Rightarrow \frac{dT}{dt}<0$ ✓

$\Rightarrow$ T has to decrease $\Rightarrow \frac{dT}{dt}<0$

$T \searrow$

if $T<A \Rightarrow T-A<0 \Rightarrow -k(T-A)>0 \Rightarrow \frac{dT}{dt}>0$

$\Rightarrow T$ will increase, ie. $T \nearrow$



Ex Population Model

$P(t)$: population size

Rate of change of population $P(t)$ wrt t is proportional to the population size itself.

$$\frac{dP}{dt} = kP, \qquad k: \text{constant} \qquad (1)$$

Note that $P(t) = Ce^{kt}$, where C is an arbitrary constant, is a solution of eq^{n} (1).

Indeed,

$$\frac{dP}{dt} = Ce^{kt} \cdot k \quad \Rightarrow \quad \frac{dP}{dt} \stackrel{?}{=} kP$$

$$Ce^{kt} \cdot k \stackrel{?}{=} \underbrace{k \cdot Ce^{kt}}_{\text{identity}} \quad ✓$$

$\therefore$ $P(t) = Ce^{kt}$ is a solution of (1).

<u>Ex</u> $P(t)$: population that satisfies $\frac{dP}{dt} = kP$

Solution is $P(t) = Ce^{kt}$.

at $t=0$ $P(0) = 1000$

at $t=1$ $P(1)$ · doubled, i.e. $P(1) = 2000$

Find population size $P(t)$ at any t.

<u>Solution</u>

We know $P(t) = Ce^{kt}$ is a solution for all t.

$P(0) = 1000$ $P(0) = Ce^{k \cdot 0} = C \Rightarrow \boxed{C = 1000}$

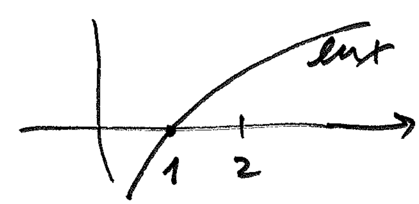
$P(t) = 1000e^{kt}$

$\underbrace{P(1)}_{\substack{\shortparallel \\ 2000}} = 1000e^{k \cdot 1} \Rightarrow 2000 = 1000e^{k}$

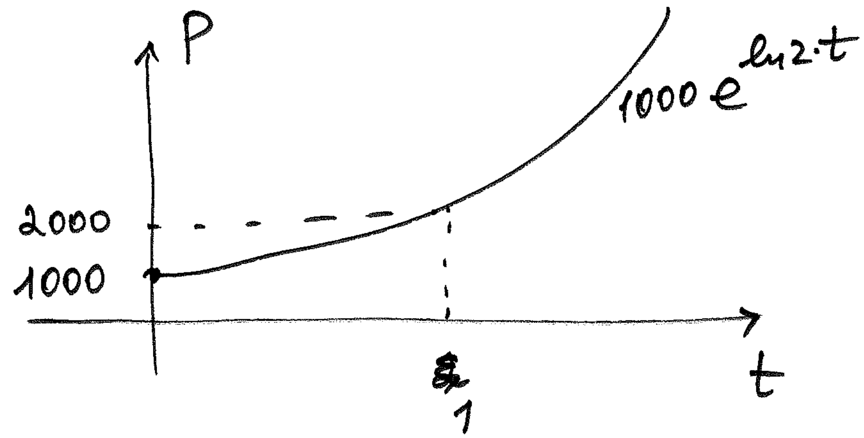
or $e^{k} = 2$ $\mid \ln$ $\ln e^{x} = x$

$\ln e^{k} = \ln 2$

$\boxed{k = \ln 2} > 0$

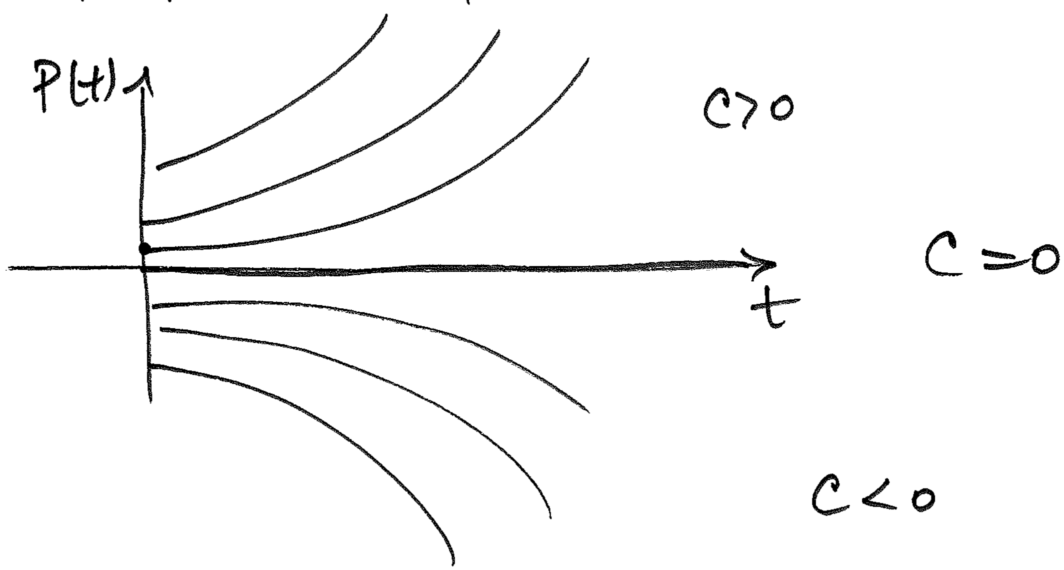


$$\boxed{P(t) = 1000\, e^{\ln 2 \cdot t}}$$



Consider again

$P(t) = C e^{kt}$, $k > 0$



$P(0) = 1000$: <u>initial condition</u>