Lecture 20

Complex roots with nonzero real part

Ex Solve $y'' + 4y' + 13y = 0$

$$(D^2 + 4D + 13)y = 0$$

$$D^2 + 4D + 13 \underset{\substack{\uparrow \\ \text{complete} \\ \text{square}}}{=} (D^2 + 4D + 4) + 9 = (D+2)^2 + 3^2$$

$$(D+2)^2 + 3^2 = 0$$

$$(D+2)^2 = -3^2$$

$$D = -2 \pm 3i \ : \ \text{roots}$$

Another way:

$$D^2 + 4D + 13 = 0$$

$$\frac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = \frac{-4 \pm 2\sqrt{4-13}}{2} = \frac{-4 \pm 2\sqrt{-9}}{2} = \frac{-4 \pm 2 \cdot i \cdot 3}{2} = -2 \pm 3i$$

same as above

$$y(x) = C_1 e^{(-2+3i)x} + C_2 e^{(-2-3i)x} = C_1 e^{-2x} \cdot e^{3ix} + C_2 e^{-2x} \cdot e^{-3ix} \ \underset{\substack{\text{Euler} \\ \text{identities}}}{=}$$

$$e^{a+b} = e^a \cdot e^b$$

$$= C_1 e^{-2x}(\cos 3x + i\sin 3x) + C_2 e^{-2x}(\cos 3x - i\sin 3x) =$$

$$= \underbrace{(C_1 + C_2)}_{K_1} e^{-2x}\cos 3x + \underbrace{i(C_1 - C_2)}_{K_2} e^{-2x}\sin 3x =$$

K_2 — complex-valued

$$= K_1 e^{-2x}\cos 3x + K_2 e^{-2x}\sin 3x$$

We can verify directly that $e^{-2x}\cos 3x$ and $e^{-2x}\sin 3x$ are also solutions of $y'' + 4y' + 13y = 0$ (and they are real-valued)

$\therefore$ $\boxed{y(x) = C_1 e^{-2x}\cos 3x + C_2 e^{-2x}\sin 3x}$ general solution

OPERATOR IDENTITY $\underline{V}$

$$\left[(D-a)^2 + b^2\right] \begin{matrix} e^{ax}\cos bx \\ e^{ax}\sin bx \end{matrix} = 0$$

$a \pm ib$

OPERATOR IDENTITY VI

$$\left[(D-a)^2 + b^2\right]^n \begin{matrix} x^k e^{ax} \cos bx \\ x^k e^{ax} \sin bx \end{matrix} = 0 \qquad k = 0, 1, \ldots, n-1$$

$$\underbrace{a \pm ib, \ldots, a \pm ib}_{n \text{ times}}$$

$-3 \pm 4i$

$e^{-3x} \cos 4x, \; e^{-3x} \sin 4x$

$y(x) = C_1 e^{-3x} \cos 4x + C_2 e^{-3x} \sin 4x$

HW on sections 3.1–3.3: due this Friday

Linear DEPENDENCE AND INDEPENDENCE. WRONSKIAN

Recall defs of hyperbolic cosine and hyperbolic sine functions:

$$\cosh(ax) = \frac{e^{ax} + e^{-ax}}{2} \qquad \sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}$$

Aside

$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

$$\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$$

$$\frac{d}{dx} \cosh(ax) = a \sinh(ax)$$

$$\frac{d}{dx} \sinh(ax) = a \cosh(ax)$$

Ex Consider DE

$$\frac{d^2y}{dx^2} - y = 0$$

$$(D^2-1)y = 0$$

± 1

$(D-a)e^{ax} = 0$

$a=1$: $e^{1\cdot x} = e^x$

$a=-1$: e^{-x}

a: root

Solutions: e^x, e^{-x}, $3e^x + 4e^{-x}$, $\cosh x$, $\sinh x$

Q Can we write the general solution as

$$y(x) = C_1 e^x + C_2 e^{-x} \ ?$$ yes

How about

$$y(x) = C_1 e^x + C_2 e^{-x} + C_3 \sinh x \ ?$$ No

$$y(x) = C_1 e^x + C_2 e^{-x} + C_3 \frac{e^x - e^{-x}}{2} = \underbrace{\left(C_1 + \frac{C_3}{2}\right)}_{=K_1} e^x + \underbrace{\left(C_2 - \frac{C_3}{2}\right)}_{=K_2} e^{-x}$$

Only two arbitrary constants, not three! We were able to reduce to two "essentially different" (linearly independent) functions.

<u>Def</u> a <u>linear combination</u> of functions $f_1(x), f_2(x), \dots, f_n(x)$ is

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

e.g. $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

<u>Def</u> a set of functions $f_1(x), f_2(x), \dots, f_n(x)$, defined on a common interval I, is said to be <u>linearly dependent</u> (LD) if there exist a set of constants $c_1, c_2, \dots, c_n$, not all being zero, such that the linear combination

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) \equiv 0 \quad \text{for all } x \in I$$

Note If at least one constant, say c_i, is non-zero, then we can write

$$f_i(x) = -\frac{c_1}{c_i} f_1(x) - \frac{c_2}{c_i} f_2(x) - \dots - \frac{c_{i-1}}{c_i} f_{i-1}(x) - \frac{c_{i+1}}{c_i} f_{i+1}(x) - \dots$$

$$\dots - \frac{c_n}{c_i} f_n(x)$$

i.e. we expressed $f_i(x)$ as a linear combination of the rest of functions.

Q Is the set of functions $\{e^x, e^{-x}, \sin x, 4\cosh x\}$ linearly dependent?

To answer this question, we have to form a linear combination of given functions and set it to zero. Then we try to find constants c_1, c_2, c_3, c_4, not all being zero, that this linear combination is zero for all x.

Thus,

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$C_1 e^x + C_2 e^{-x} + C_3 \sin x + C_4 \cdot 4 \cosh x = 0$$

$$C_1 e^x + C_2 e^{-x} + C_3 \sin x + C_4 \cdot 2 (e^x + e^{-x}) = 0$$

$$\underbrace{(C_1 + 2C_4)}_{=0} e^x + \underbrace{(C_2 + 2C_4)}_{=0} e^{-x} + \underbrace{C_3}_{=0} \sin x = 0$$

$$C_1^2 + C_2^2 + C_3^2 + C_4^2 \neq 0$$

$C_1 + 2C_4 = 0$ $\quad$ let $\boxed{C_4 = 1} \Rightarrow \boxed{C_1 = -2}$

$C_2 + 2C_4 = 0$ $\quad$ $\boxed{C_2 = -2}$

$\boxed{C_3 = 0}$

Since we were able to find a set of constants C_1, C_2, C_3, C_4, not all being zero, functions $\{e^x, e^{-x}, \sin x, 4\cosh x\}$ are LD.

<u>Note</u> The choice of constants is not unique.

$$C_4 = 2, \quad C_1 = -4, \quad C_2 = -4, \quad C_3 = 0$$

$$C_4 = -\tfrac{1}{2}, \quad C_1 = 1, \quad C_2 = 1, \quad C_3 = 0$$

<u>Def</u> A set of functions is <u>linearly independent</u> (LI) if they are not linearly dependent.

To show that functions are linearly independent we must show that

$$C_1 f_1(x) + C_2 f_2(x) + \dots + C_n f_n(x) \equiv 0 \quad \text{for all } x \in I$$

implies

$$C_1 = C_2 = \dots = C_n = 0.$$

<u>Q</u> Is the set of functions LD or LI?

$$\{e^x, xe^x, x\}$$

$C_1 e^x + C_2 x e^x + C_3 x \equiv 0$ for all x

Since this is true for all x, it has to be true at $x=0$

$x=0$: $C_1 e^0 + C_2 \cdot \cancel{0}^{0} e^0 + C_3 \cdot \cancel{0}^{0} = 0 \Rightarrow C_1 = 0$

$C_2 x e^x + C_3 x = 0 \quad \Big| \ \frac{1}{x} \qquad x \neq 0$

$C_2 e^x + C_3 = 0$ — true for all $x \neq 0$

$x=1$: $C_2 e^1 + C_3 = 0$

$x=-1$: $C_2 e^{-1} + C_3 = 0 \quad \Big| -$ $\qquad C_2 \underbrace{\left(e - \frac{1}{e}\right)}_{\neq 0} = 0 \Rightarrow C_2 = 0$

$C_3 = -C_2 e^1 = 0$

$\Rightarrow C_1 = C_2 = C_3 = 0$

$\Rightarrow \{e^x, x e^x, x\}$ is linearly independent.