Lecture 21

Q Is the set of functions LD or LI?

$\{e^x, 2x, x-e^x\}$

$C_1 e^x + C_2 \cdot 2x + C_3 (x - e^x) = 0$ for all x

$\underbrace{(C_1 - C_3)}_{=0} e^x + \underbrace{(2C_2 + C_3)}_{=0} x = 0$

$\begin{cases} C_1 - C_3 = 0 \\ 2C_2 + C_3 = 0 \end{cases}$ two equations for 3 unknowns C_1, C_2, C_3

Let C_3 be a free parameter.

Let $C_3 = 1$. Then $C_1 = C_3 = 1$

$2C_2 = -C_3 \Rightarrow C_2 = -\frac{1}{2} C_3 = -\frac{1}{2}$

$\therefore \quad 1 \cdot e^x - \frac{1}{2} \cdot 2x + 1(x - e^x) = 0$: nontrivial linear combination

Since we found a nontrivial linear combination of $e^x, 2x, x-e^x$, these functions are LD.

__Thm__ Given n^{th} order linear DE w/ constant coefficients, there exist n linearly independent solutions of this equation. The general solution is their linear combination.

__Ex__ $y''-y=0$ has solutions $e^x, e^{-x}, \cosh x, \sinh x$.

Possible general solutions:

$y(x) = C_1 e^x + C_2 e^{-x}$

$y(x) = C_1 e^x + C_2 \cosh x$

$y(x) = C_1 \cosh x + C_2 \sinh x$

etc.

WRONSKIAN

Consider a set of three functions $\{f_1(x), f_2(x), f_3(x)\}$.
Assume that these functions have first and second derivatives.
Are these functions LD or LI?

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \Big| \frac{d}{dx}$$

$$c_1 f_1'(x) + c_2 f_2'(x) + c_3 f_3'(x) = 0 \quad \Big| \frac{d}{dx}$$

$$c_1 f_1''(x) + c_2 f_2''(x) + c_3 f_3''(x) = 0$$

We have 3 equations for 3 unknowns c_1, c_2, c_3. We can write this system in a matrix form.

$$\begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underset{\text{RHS}}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

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CRAMER'S RULE

$$\Delta = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = W(f_1, f_2, f_3) = W(x):$$

Wronskian of $f_1(x),\ f_2(x),\ f_3(x)$

$$\Delta_1 = \begin{vmatrix} 0 & f_2 & f_3 \\ 0 & f_2' & f_3' \\ 0 & f_2'' & f_3'' \end{vmatrix} = 0$$

RHS vector

$$\Delta_2 = \begin{vmatrix} f_1 & 0 & f_3 \\ f_1' & 0 & f_3' \\ f_1'' & 0 & f_3'' \end{vmatrix} = 0$$

RHS vector

$$\Delta_3 = \begin{vmatrix} f_1 & f_2 & 0 \\ f_1' & f_2' & 0 \\ f_1'' & f_2'' & 0 \end{vmatrix} = 0$$

RHS vector

Then

$$C_1 = \frac{\Delta_1}{\Delta}, \quad C_2 = \frac{\Delta_2}{\Delta}, \quad C_3 = \frac{\Delta_3}{\Delta}$$

or $$C_1 = \frac{0}{W}, \quad C_2 = \frac{0}{W}, \quad C_3 = \frac{0}{W}$$

If $W \neq 0 \Rightarrow C_1 = C_2 = C_3 = 0$

We wrote

$$C_1 f_1(x) + C_2 f_2(x) + C_3 f_3(x) = 0$$

and if $W \neq 0$, we obtained that $C_1 = C_2 = C_3 = 0$.

$\Rightarrow f_1(x), f_2(x), f_3(x)$ are LI.

If $W = 0 \Rightarrow$ NO INFO (in general)

Ex $\{e^x, \frac{1}{x}, 1\}$ LD or LI ?, $x > 0$

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Recall how to compute determinant of 3x3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \underbrace{(-1)^{2+1}}_{=-1} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \cdot \underbrace{(-1)^{2+2}}_{=1} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} +$$

$$+ a_{23} \cdot \underbrace{(-1)^{2+3}}_{=-1} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

a_{ij} : i: row, j: column

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Back to example: $\{e^x, \frac{1}{x}, 1\}$ LD or LI ?

$$W = \begin{vmatrix} e^x & \frac{1}{x} & \boxed{1} \\ e^x & -\frac{1}{x^2} & 0 \\ e^x & \frac{2}{x^3} & 0 \end{vmatrix} = \boxed{1} \cdot (-1)^{1+3} \cdot \begin{vmatrix} e^x & -\frac{1}{x^2} \\ e^x & \frac{2}{x^3} \end{vmatrix} + \cancel{0 \cdot (-1)^{2+3} \begin{vmatrix} e^x & \frac{1}{x} \\ e^x & \frac{2}{x^3} \end{vmatrix}} +$$

0

$$\cancel{+0(-1)^{3+3}\begin{vmatrix} e^x & \frac{1}{x} \\ e^x & -\frac{1}{x^2} \end{vmatrix}}^{\to 0} = e^x \cdot \frac{2}{x^3} + e^x \cdot \frac{1}{x^2} = \underbrace{e^x}_{\neq 0} \underbrace{\left(\frac{2}{x^3} + \frac{1}{x^2}\right)}_{\substack{>0 \\ x>0}} \neq 0$$

$W \neq 0 \Rightarrow$ functions $\{e^x, \frac{1}{x}, 1\}$ are LI.

Ex $\{x^2, x|x|\}$ LI or LD?

for $x \geq 0 \Rightarrow |x| = x \Rightarrow \{x^2, x^2\}$

$$W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$$

for $x < 0 \Rightarrow |x| = -x \Rightarrow \{x^2, -x^2\}$

$$W = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$$

NO INFO

One can show using def that $x^2, x|x|$ are LI.

Ex $\{e^x, 2e^x\}$ LD or LI ?

$$W = \begin{vmatrix} e^x & 2e^x \\ e^x & 2e^x \end{vmatrix} = 0$$

$f_1 = e^x, \quad f_2 = 2e^x = 2f_1 \Rightarrow 2f_1 - f_2 = 0$

$\therefore$ $e^x, 2e^x$ are LD.

While $W=0$ gives no information to whether a set of functions is LD or LI, there is a special case when the functions are solutions of a linear DE and we have:

Thm If the set of functions $\{f_1, f_2, .., f_n\}$ is a solution set of a linear DE, then

$W \neq 0 \Rightarrow$ LI

$W = 0 \Rightarrow$ LD

An application of LD, LI

Ex Solve

$$K_1 e^x + K_2 e^{-x} + K_3 \sin x = 4e^x - 3e^{-x} + 2\sin x$$

to find K_1, K_2, K_3.

We can write

$$(K_1 - 4)e^x + (K_2 + 3)e^{-x} + (K_3 - 2)\sin x = 0$$

Functions $\{e^x, e^{-x}, \sin x\}$ are LI

$$\Rightarrow \quad K_1 - 4 = 0 \qquad K_2 + 3 = 0 \qquad K_3 - 2 = 0$$

or $\quad K_1 = 4 \qquad K_2 = -3 \qquad K_3 = 2$

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Complex Numbers

$z = x + iy$: complex number

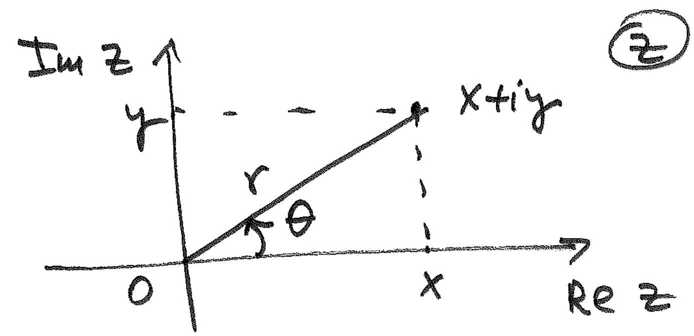
$x = \mathrm{Re}\{z\}$: real part of z

$y = \mathrm{Im}\{z\}$: imaginary part of z

x, y : real #s

$\underbrace{z = x + iy}_{\text{Cartesian representation}} = re^{i\theta}$: polar form

$i^2 = -1$



$0 \le \theta < 2\pi$

Euler's identity

$e^{i\theta} = \cos\theta + i\sin\theta$

$z = \underline{\underline{x}} + iy = re^{i\theta} = r(\cos\theta + i\sin\theta) = \underline{\underline{r\cos\theta}} + ir\sin\theta$

$\therefore \quad x = r\cos\theta$

$y = r\sin\theta$