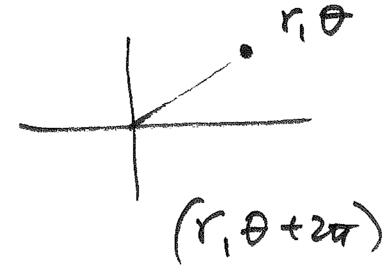
Lecture 22

Complex Numbers (Cont'd)

Q If we have Cartesian coordinates (x,y), how do find polar coordinates (ρ, θ)?



$x = r\cos\theta, \quad y = r\sin\theta$

$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) = r^2$

$\Rightarrow \boxed{r^2 = x^2 + y^2} \Rightarrow \boxed{r = \sqrt{x^2+y^2}}$

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \boxed{\frac{y}{x} = \tan\theta}$

II: $\theta = \arctan\frac{y}{x} + \pi$, $x<0, y>0$

I: $\theta = \arctan\frac{y}{x}$, $x>0, y>0$

III: $\theta = \arctan\frac{y}{x} + \pi$, $x, y < 0$

IV: $\theta = \arctan\frac{y}{x} + 2\pi$, $x>0, y<0$

$0 \le \theta < 2\pi$

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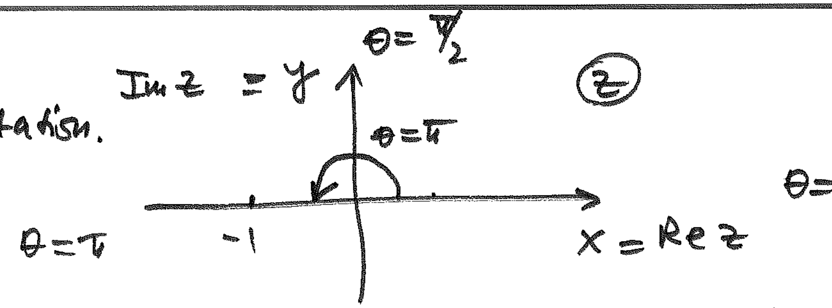
$\underline{\underline{\text{Ex}}}$ $z = -1$ Find polar representation.



$\theta = \frac{3\pi}{2}$

$z = re^{i\theta}$, $r - ?$, $\theta - ?$

$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+0^2} = 1$

$-1 = \underbrace{-1}_{= x} + \underbrace{0}_{y} \cdot i$

$x = -1, \ y = 0$

$\Rightarrow \ -1 = 1 \cdot e^{i\pi}$ or $\boxed{-1 = e^{i\pi}}$

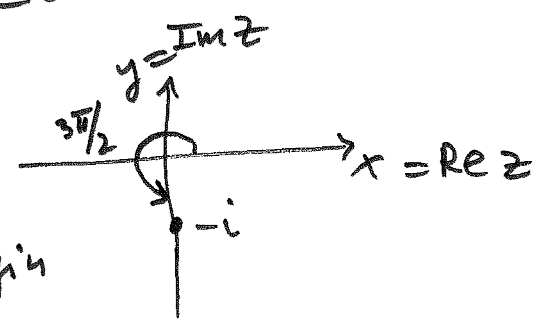
polar representation of $z = -1$

$e^{i\pi} = \underbrace{\cos\pi}_{-1} + i\cancel{\sin\pi}^{\,0} = -1$ ✓

r: magnitude

θ: angle, $0 \le \theta < 2\pi$

$\underline{\underline{\text{Ex}}}$ $z = -i$



$r = 1$: distance from origin

$z = -i = 0 + (-1) \cdot i$

$x = 0, \ y = -1$

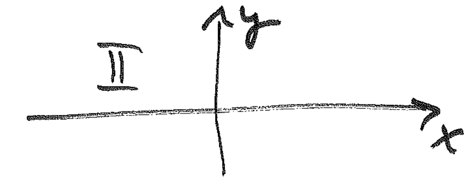
$r = \sqrt{x^2+y^2} = \sqrt{0^2+(-1)^2} = 1$

$\theta = \frac{3\pi}{2}$

$\Rightarrow z = -i = re^{i\theta} = 1 \cdot e^{\frac{3\pi}{2} i}$

$\Rightarrow \boxed{-i = e^{\frac{3\pi}{2} i}}$

Ex $z = -1 + \sqrt{3}\, i \qquad x = -1, \quad y = \sqrt{3}$



$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \boxed{2 = r}$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1}$

pt is in II quadrant $\Rightarrow$ $\theta = \pi + \arctan\frac{y}{x} = \pi + \arctan\left(\frac{\sqrt{3}}{-1}\right) =$

arctan t is odd function, i.e. $\arctan(-t) = -\arctan t$

$= \pi - \arctan\sqrt{3} \; \boxed{=}$

$\boxed{\sin\frac{\pi}{6} = \frac{1}{2}} \longrightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan\frac{\pi}{6} = \frac{\sin\pi/6}{\cos\pi/6} = \frac{1}{\sqrt{3}}$

$\downarrow \qquad \searrow \qquad \Downarrow$

$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \tan\frac{\pi}{3} = \sqrt{3}$

$\boxed{=}\; \pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3} = \theta} \in [0, 2\pi) \qquad \therefore \quad \boxed{-1 + \sqrt{3}\, i = 2e^{i\frac{2\pi}{3}}}$

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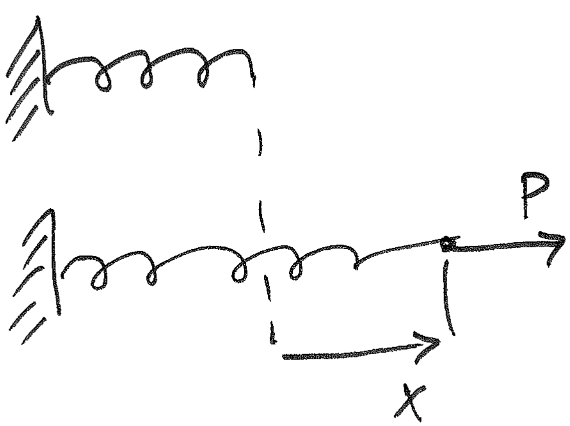
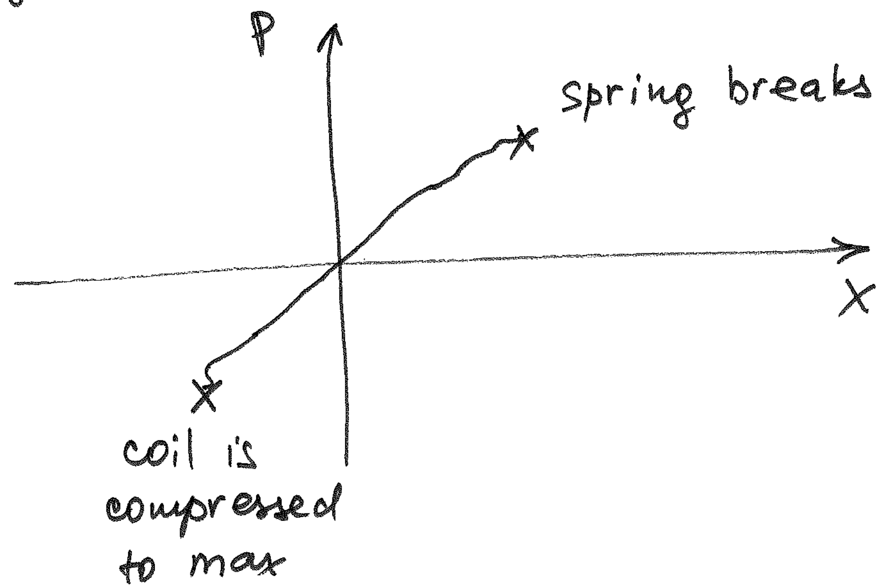
3.4 Applications of linear DEs w/ constant coefficients (homogeneous). Mechanical systems.

Experiment: consider a spring attached to a wall



Let P be a force (in pounds) applied to a spring. Let x be a resulting stretching (or compression).



Experimentally we observe that force is smooth function for small x : $-a < x < a$.

Expand $P(x)$ in a Taylor series about $x=0$:

$$P(x) = \underbrace{\cancel{P(0)}^{0} + P'(0)\cdot x}_{} + \frac{P''(0)}{2!}x^2 + \dots$$

But $P(0)=0$. If we keep only a linear part $P(0)+P'(0)\cdot x$, we get a good approximation of $P(x)$ for small x.

Denote $k = P'(0)$.

$$\Rightarrow \quad P(x) \approx P'(0)\cdot x = kx$$

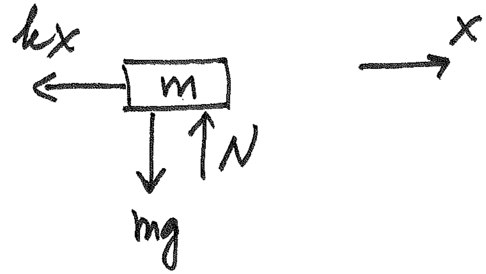
This is known as Hooke's law:

$k>0$: spring constant $\qquad \boxed{P(x) = kx}$

Place an object of mass m on a frictionless support.

If we pull mass x units to the right, the spring will exert force kx directed to the left (restoring force)

Consider FBD



<u>Note</u> If $x>0$ (mass is pulled to the right), force acts to the left.
If $x<0$ (mass is pulled to the left), force will act to the right.

Adding forces in horizontal direction and applying Newton's 2nd law we get

x: displacement from equilibrium position

$ma = F$

$\dot{x} = \frac{dx}{dt} = v$: velocity

$m\frac{d^2x}{dt^2} = -kx$

$\ddot{x} = \frac{d^2x}{dt^2} = a$: acceleration

ICs:

$x(0)=x_0$: initial displacement

$\dot{x}(0) = v_0$: initial velocity

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \Big| \frac{1}{m}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$k > 0$: spring const

$$\left(D^2 + \frac{k}{m}\right)x = 0$$

$m > 0$: mass

$\omega_0^2 = \frac{k}{m}$; ω_0: natural frequency

$$\left(D^2 + \omega_0^2\right)x = 0$$

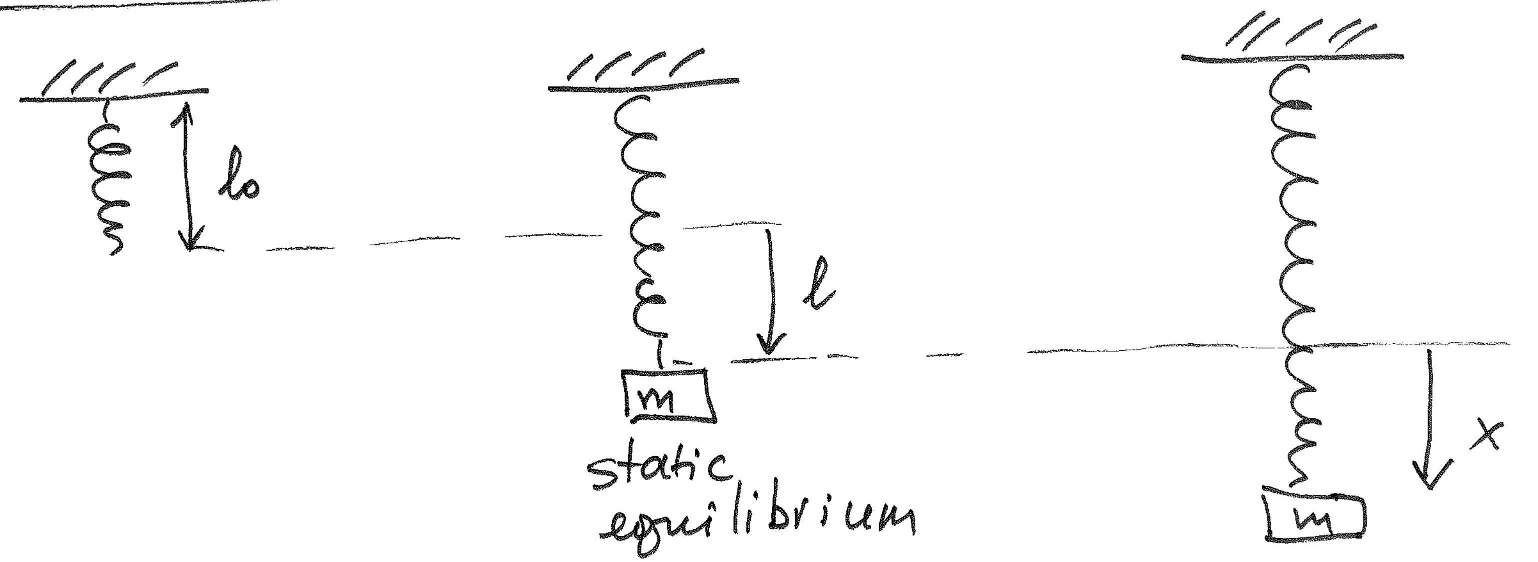
$\omega_0 = \sqrt{\frac{k}{m}}$ $\left[\frac{rad}{sec}\right]$

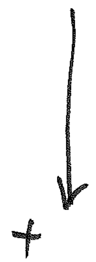
$\pm i\omega_0$

$D^2 = -\omega_0^2$

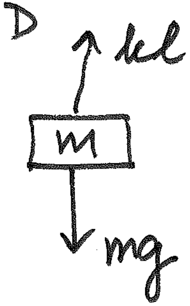
$D = \pm i\omega_0$

$\boxed{x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}$: simple harmonic motion

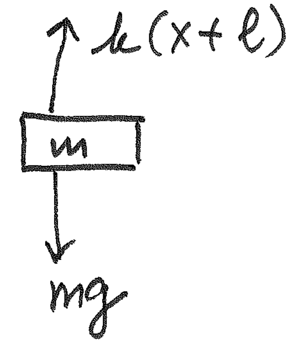




FBD



FBD



$mg = kl$
from here we can find k

Apply Newton's 2nd law:

$$m\frac{d^2x}{dt^2} = \cancel{mg} - k(x+\cancel{l})$$

From static equilibrium: $mg = kl$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -kx$$: the same equation as for horizontal motion!