Lecture 23

Now the positive direction is upward direction.

$+\uparrow$ Let's choose $y = -x$. Then

$$\frac{dy}{dt} = -\frac{dx}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = -\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt} = -kx$$

$$\Rightarrow \quad m\left(-\frac{d^2y}{dt^2}\right) = -k(-y) \quad \Big| \cdot (-1)$$

$m\dfrac{d^2y}{dt^2} = -ky$: the same equation as before!

ICs: $y(t_0) = y_0$

$y'(t_0) = v_0$

Consider again

$$m\frac{d^2x}{dt^2} = -kx$$

$$m\frac{d^2x}{dt^2} + kx = 0, \qquad x(0) = x_0, \qquad \frac{dx}{dt}(0) = v_0$$

$$(mD^2 + k)x = 0$$

$$m\left(D^2 + \frac{k}{m}\right)x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}} : \text{ natural frequency}$$

$m \neq 0$

$$(D^2 + \omega_0^2)x = 0$$

$$\pm i\omega_0$$

$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$: simple harmonic motion

Using IC, we can find C_1, C_2.

$$x(t) = A\cos(\omega_0 t - \delta)$$

A: amplitude $\quad A = \sqrt{C_1^2 + C_2^2}$

δ: phase / phase shift / time lag $\quad \tan\delta = \frac{C_2}{C_1}$

$T = \frac{2\pi}{\omega_0}$: period of oscillations

$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ [# of cycles per sec or Hertz]

Recall $\quad \cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y \qquad (*)$

$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t = \sqrt{C_1^2+C_2^2}\left(\underbrace{\frac{C_1}{\sqrt{C_1^2+C_2^2}}}_{=\cos\delta}\cos\omega_0 t + \underbrace{\frac{C_2}{\sqrt{C_1^2+C_2^2}}}_{=\sin\delta}\sin\omega_0 t\right)$$

$$\overset{(*)}{=} \sqrt{C_1^2+C_2^2}\,\cos(\delta - \omega_0 t) \underset{\substack{\text{even} \\ f^{n} \\ \cos(-t)=\cos t}}{\overset{\cos\text{ is}}{=}} \underbrace{\sqrt{C_1^2+C_2^2}}_{=A}\cos(\omega_0 t - \delta) = A\cos(\omega_0 t - \delta)$$

Check: $\cos^2\delta + \sin^2\delta \overset{?}{=} 1$

$$\cos^2\delta + \sin^2\delta = \left(\frac{C_1}{\sqrt{C_1^2+C_2^2}}\right)^2 + \left(\frac{C_2}{\sqrt{C_1^2+C_2^2}}\right)^2 = \frac{C_1^2}{C_1^2+C_2^2} + \frac{C_2^2}{C_1^2+C_2^2} = 1$$

We denoted

$$\cos\delta = \frac{C_1}{\sqrt{C_1^2+C_2^2}}, \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2+C_2^2}}$$

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{C_2/\sqrt{C_1^2+C_2^2}}{C_1/\sqrt{C_1^2+C_2^2}} = \frac{C_2}{C_1} \Rightarrow \boxed{\tan\delta = \frac{C_2}{C_1}}$$

C_2

$\delta = \pi + \arctan\frac{C_2}{C_1}$

$C_1 < 0,\ C_2 > 0$

$\delta = \arctan\frac{C_2}{C_1}$

$C_1, C_2 > 0$

C_1

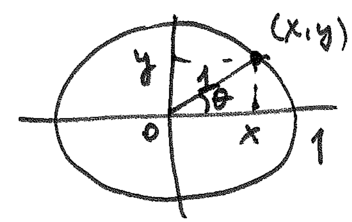
$\delta = \pi + \arctan\frac{C_2}{C_1}$

$C_1, C_2 < 0$

$\delta = 2\pi + \arctan\frac{C_2}{C_1}$

$C_1 > 0,\ C_2 < 0$

$0 \le \delta < 2\pi$



$$\cos\theta = \frac{x}{1} = x$$

$$\sin\theta = \frac{y}{1} = y$$

Energy

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \Big| \cdot \frac{dx}{dt}$$

$x = x(t)$

$$m\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + kx\underbrace{\frac{dx}{dt}} = 0$$

$$\frac{d}{dt}\left[x^2\right] = 2x\cdot\underbrace{\frac{dx}{dt}}$$

$$\frac{m}{2}\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2\right] + \frac{k}{2}\frac{d}{dt}\left[x^2\right] = 0$$

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2\right] = 2\cdot\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}$$

$$\frac{d}{dt}\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2\right] = 0$$

$$\frac{k}{2}\frac{d}{dt}\left[x^2\right] = \frac{k}{2}\cdot 2x\frac{dx}{dt}$$

$$= kx\cdot\frac{dx}{dt}$$

$$\frac{d}{dt}\left[\frac{m}{2}v^2 + \frac{k}{2}x^2\right] = 0$$

$$\underbrace{\frac{m}{2}v^2}_{\text{kinetic energy}} + \underbrace{\frac{k}{2}x^2}_{\text{potential energy}} = \text{const}$$ – does not depend on time t

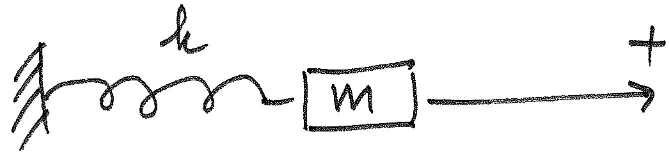
$\frac{m}{2} v^2 + \frac{k}{2} x^2 = E$ – the same for all times, including at $t=0$

total energy

$\Rightarrow E = \frac{m}{2} v_0^2 + \frac{k}{2} x_0^2$

$\Rightarrow \boxed{\frac{m}{2} v^2 + \frac{k}{2} x^2 = \frac{m}{2} v_0^2 + \frac{k}{2} x_0^2}$ conservation of energy

<u>Problem</u> A mass of $\frac{1}{16}$ slug is attached to a spring whose spring constant is 4 lb/ft. Initially the mass is $\frac{2}{3}$ ft to the right of equilibrium and is moving to the left with velocity of $\frac{4}{3}$ ft/sec. Determine $x(t)$ which describes the subsequent motion. When does the mass reach maximum displacement for the first time?



$k = 4$ lb/ft

$m = \frac{1}{16}$ slug

$x(0) = \frac{2}{3}$ ft (to right)

$\dot{x}(0) = v_0 = -\frac{4}{3}$ ft/s (to left)

$m\ddot{x} + kx = 0$

to the right

moving to the left

$\frac{1}{16}\ddot{x} + 4x = 0 \quad \Big| \cdot 16 \qquad x(0) = \frac{2}{3}, \quad \dot{x}(0) = -\frac{4}{3}$

$\ddot{x} + 64x = 0$

$(D^2 + 64)x = 0$

$\pm i8 \qquad \omega_0 = 8 = \sqrt{\frac{k}{m}}$

$x(t) = C_1 \cos 8t + C_2 \sin 8t$

To find C_1, C_2 we use ICs.

$\underbrace{x(0)}_{= \frac{2}{3}} = C_1 \underbrace{\cos 0}_{=1} + C_2 \cancelto{0}{\sin 0} \Rightarrow C_1 = \frac{2}{3}$

$\dot{x}(t) = -8C_1 \sin 8t + C_2 \cdot 8 \cos 8t$

$\underbrace{\dot{x}(0)}_{= -\frac{4}{3}} = -8C_1 \cancelto{0}{\sin 0} + 8C_2 \underbrace{\cos 0}_{=1} \Rightarrow C_2 = -\frac{1}{6}$

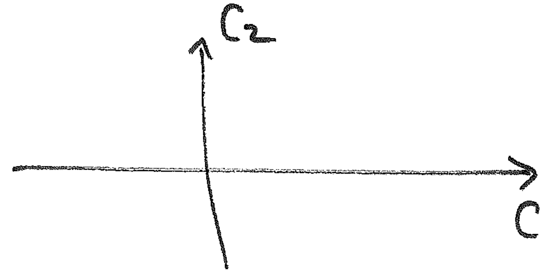
$$\boxed{x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t}$$

$x(t) = A\cos(\omega_0 t - \delta)$ $\qquad \omega_0 = 8$

$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \boxed{\frac{\sqrt{17}}{6} = A}$$

δ: phase shift

$\tan\delta = \frac{C_2}{C_1}$



$C_1 = \frac{2}{3} > 0, \quad C_2 = -\frac{1}{6} < 0$

$-\frac{1}{6}\cdot\frac{3}{2} = -\frac{1}{4}$

$\underline{IV}$ quadrant

$$\delta = 2\pi + \arctan\left(\frac{C_2}{C_1}\right) = 2\pi + \arctan\left(\frac{-\frac{1}{6}}{\frac{2}{3}}\right) = 2\pi + \arctan\left(-\frac{1}{4}\right) =$$

odd f^n

$$= 2\pi - \arctan\frac{1}{4}$$

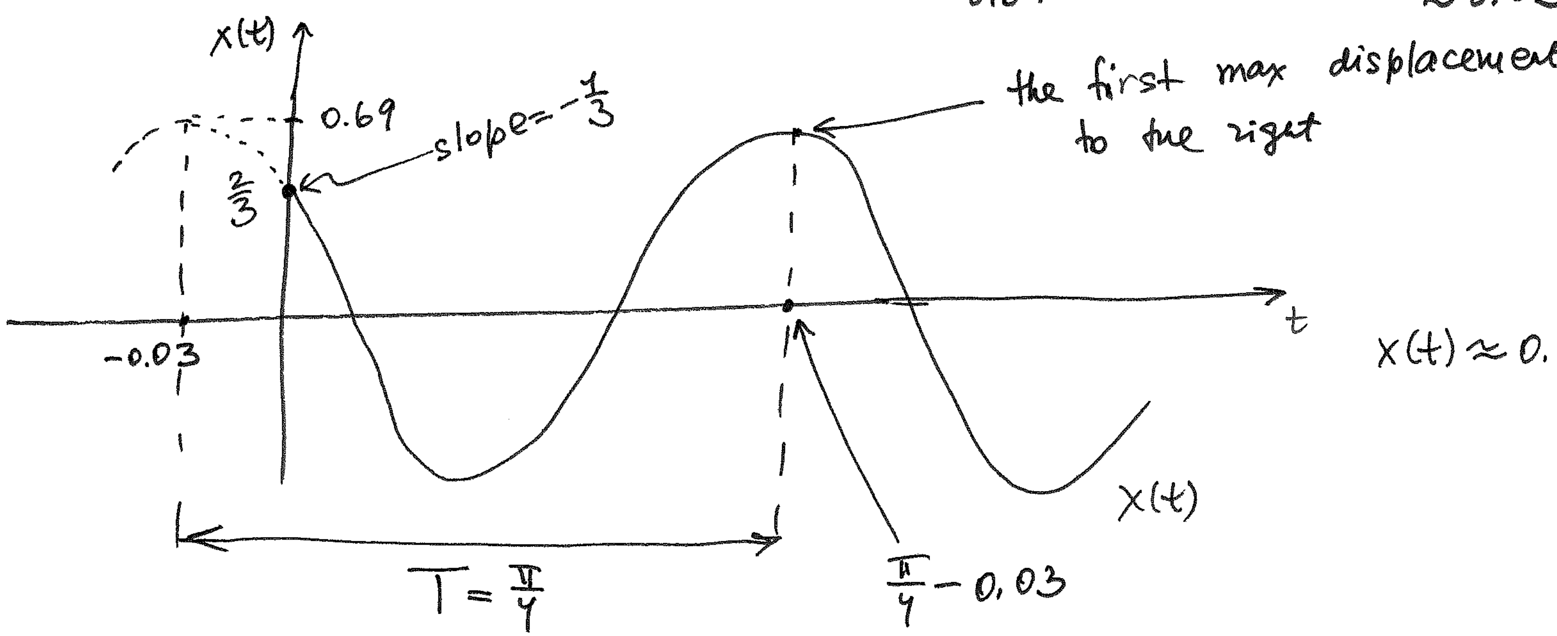
$$\Rightarrow \boxed{\delta = 2\pi - \arctan\frac{1}{4}}$$

Hence,

$$x(t) = A\cos(\omega_0 t - \delta) = \frac{\sqrt{17}}{6}\cos\left(8t - \left[2\pi - \arctan\frac{1}{4}\right]\right) \;=$$

cosine is a 2π-periodic function

$$= \frac{\sqrt{17}}{6}\cos\Big(8t + \underbrace{\arctan\frac{1}{4}}_{\approx 0.245}\Big) = \underbrace{\frac{\sqrt{17}}{6}}_{\approx 0.69}\cos\Big[8\Big(t + \underbrace{\frac{\arctan\frac{1}{4}}{8}}_{\approx 0.03}\Big)\Big]$$



$x(t) \approx 0.69\cos[8(t + 0.03)]$

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{8} = \frac{\pi}{4}$$