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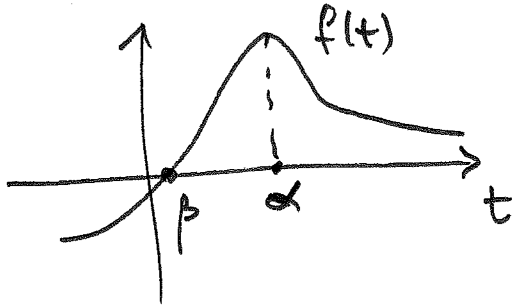
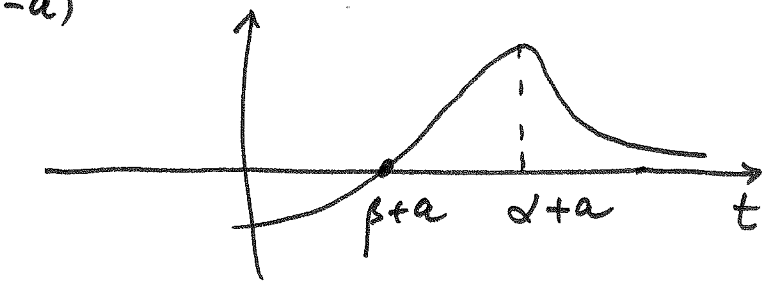
Lecture 24



$f(t-a)$



To get graph of $f(t-a)$, we shift graph of $f(t)$ by a units to the right.

$x(t) \approx 0.69 \cos[8(t+0.03)]$ ← see previous Lecture (#23)

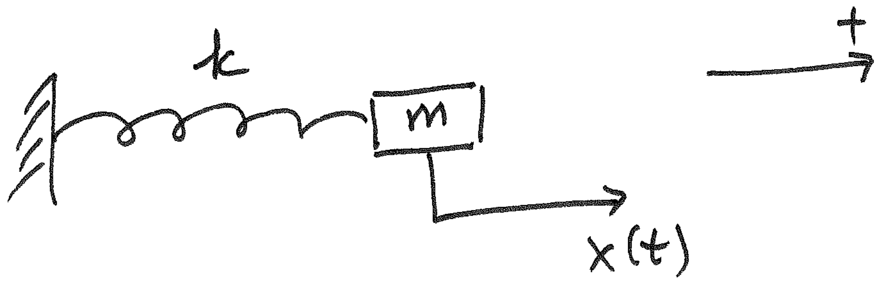
From the graph of $x(t)$ we see that max displacement to to right occurs for the first time at $t = -0.03 + \frac{\pi}{4} = 0.755$ (s)

<u>Note</u> alternatively, to find the time when we have max displacement to the right we solve $\cos[8(t+0.03)] = 1$ for t and find the smallest $t > 0$.

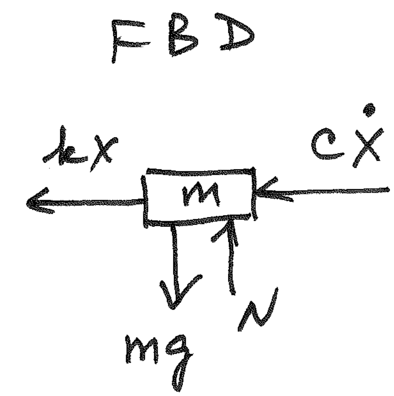
<u>Note</u> If we need to find time when mass crosses equilibrium position for the first time, we would solve $x(t) = 0$ for smallest t, i.e. $0.69\cos[8(t+0.03)] = 0 \Rightarrow \cos[8(t+0.03)] = 0$
$\Rightarrow$ solve for t

DAMPED MOTION

Suppose that in the previous model there is an additional force, a damping, which is proportional to velocity and always opposite in sign to the velocity vector. Let c be a proportionality constant (c is a damping coefficient).
$c>0$



$x(t)$: displacement from equilibrium position



Newton's 2nd law: $m\ddot{x} = -kx - c\dot{x}$

$m\ddot{x} + c\dot{x} + kx = 0$, $\quad x(0) = x_0$, $\dot{x}(0) = v_0$

$$(mD^2 + cD + k)\,x = 0$$

roots: $\dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

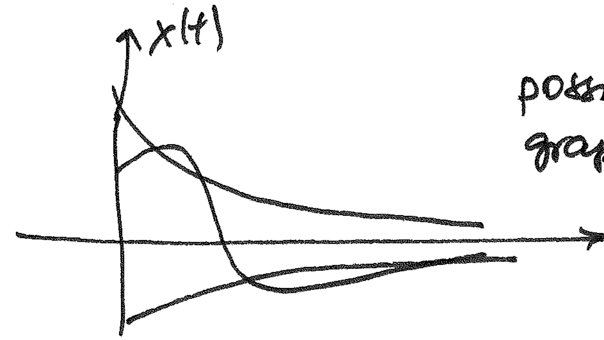
Whether the roots are real distinct, real repeated or complex conjugate depends on the sign of discriminant $c^2 - 4mk$.

<u>Case 1</u> OVERDAMPED CASE $c^2 - 4mk > 0$ or $c^2 > 4mk$

2 real distinct roots (negative)

$$\underbrace{\frac{-c + \sqrt{c^2 - 4mk}}{2m}}_{\equiv -p_1} < 0 \qquad \underbrace{\frac{-c - \sqrt{c^2 - 4mk}}{2m}}_{\equiv -p_2} < 0$$

Then $\boxed{x(t) = C_1 e^{-p_1 t} + C_2 e^{-p_2 t}}$



To find C_1, C_2 we use ICs:

$x(0) = x_0, \quad \dot{x}(0) = v_0$

<u>Note</u>: mass can cross equilibrium at most once

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<u>Case 2</u> CRITICALLY DAMPED $c^2 - 4mk = 0$ or $c^2 = 4mk$

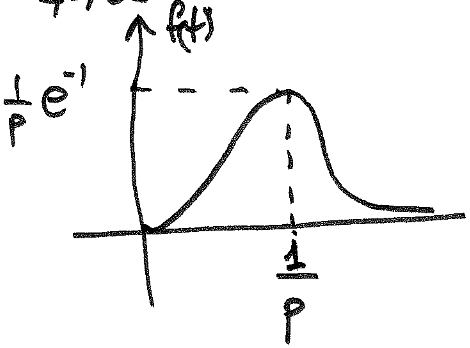
2 real repeated roots $\underbrace{-\frac{c}{2m}}_{-p}, \underbrace{-\frac{c}{2m}}_{-p}$

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}$$

~~let~~ $f(t) = te^{-pt}$. What is $\lim\limits_{t\to\infty} f(t)$?

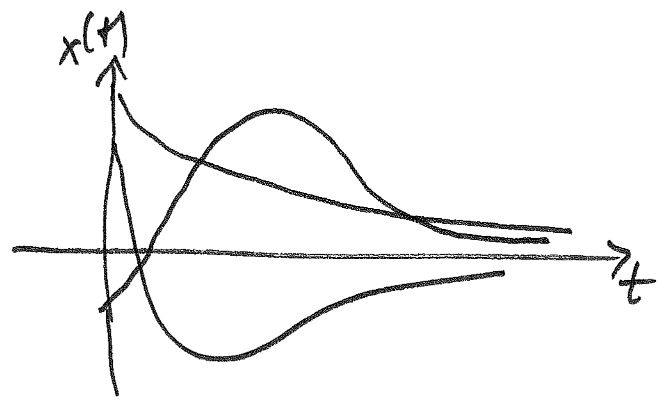
$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} te^{-pt} = \infty \cdot 0 = \lim_{t\to\infty} \frac{t}{e^{pt}} \overset{\text{l'Hopital}}{\underset{\text{rule}}{=}} \lim_{t\to\infty} \frac{1}{pe^{pt}} = 0$$

$\frac{\infty}{\infty}$



$$\frac{df}{dt} = e^{-pt}(1-pt)$$

Possible graphs of $x(t)$



Mass cannot cross equilibrium position ($x(t)=0$) more than once.

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Case 3 Underdamped or Oscillatory Motion

$c^2 - 4mk < 0$ or $c^2 < 4mk$

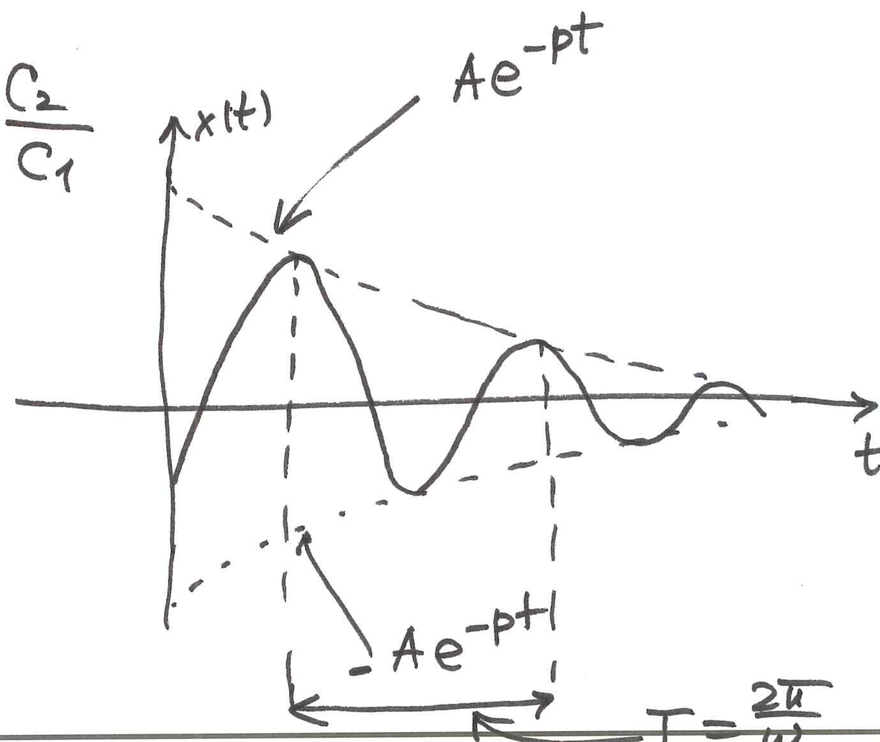
$\dfrac{-c \pm i\sqrt{4mk - c^2}}{2m}$ or $-p \pm i\omega$

Solution:

$x(t) = C_1 e^{-pt}\cos\omega t + C_2 e^{-pt}\sin\omega t = e^{-pt}\left(C_1\cos\omega t + C_2\sin\omega t\right) =$

$= \underbrace{Ae^{-pt}}\cos(\omega t - \alpha)$

where $A = \sqrt{C_1^2 + C_2^2}$, $\tan\alpha = \dfrac{C_2}{C_1}$

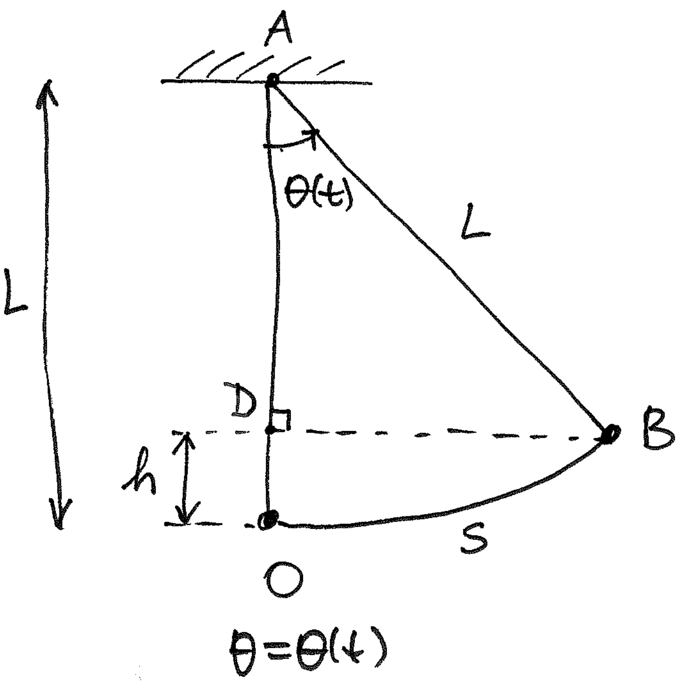


Ae^{-pt}: time-varying amplitude

$T = \frac{2\pi}{\omega}$: pseudoperiod or quasiperiod

ω: pseudofrequency

The Simple Pendulum



Consider a weightless rod of length L. The mass m is attached to one of its ends.

We will use conservation of energy to derive DE for $\theta(t)$.

The total energy is constant.

Total energy = Kinetic Energy + Potential Energy

Arc OB has length $s = L\theta$

Then velocity of mass m is $\dot{s} = L\dot{\theta}$ or $v = \frac{ds}{dt} = L\frac{d\theta}{dt}$

Kinetic Energy $= KE = \frac{mv^2}{2} = \frac{m}{2}\left(L\frac{d\theta}{dt}\right)^2$

Potential Energy $= PE = mgh$

From Δ_{ADB} : $\cos\theta = \frac{AD}{AB} \Rightarrow AD = AB\cdot\cos\theta = L\cos\theta$

Then $h = OD = OA - AD = L - L\cos\theta = L(1-\cos\theta)$

$\therefore \quad PE = mgh = mgL(1-\cos\theta)$

KE+PE=const

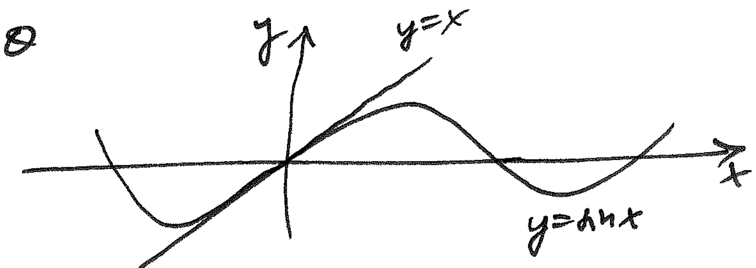
$$\underbrace{\frac{m}{2}L^2\left(\frac{d\theta}{dt}\right)^2}_{KE} + \underbrace{mgL(1-\cos\theta)}_{PE} = \text{const} \quad \Big| \ \frac{d}{dt} \qquad \theta=\theta(t)$$

$$\frac{m}{2}L^2\cdot 2\,\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + mgL\cdot\sin\theta\cdot\frac{d\theta}{dt} = 0 \quad \Big| \ \frac{1}{\frac{d\theta}{dt}\,mL^2}, \quad \frac{d\theta}{dt}\neq 0$$

$\therefore$ $\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0}$ simple nonlinear pendulum equation

linearize $\sin\theta$ by expanding it in Taylor series about $\theta=0$:

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$ for small θ



$\Rightarrow$ $\boxed{\sin\theta \approx \theta \quad \text{for small } \theta}$

Then $\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0}$ linearized pendulum equation

$\omega^2 = \frac{g}{L}$ $\quad \omega = \sqrt{\frac{g}{L}}$: natural frequency

$\theta(t) = C_1\cos\sqrt{\frac{g}{L}}\,t + C_2\sin\sqrt{\frac{g}{L}}\,t = A\cos\left(\sqrt{\frac{g}{L}}\,t - \alpha\right)$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{L}}} = \sqrt{\frac{L}{g}}\,2\pi$

Note If T, L are known, we can compute g. This is one of the ways to compute g (on Mars, for example).