Lecture 25

#41 S3.3

Find a linear homogeneous constant-coefficient equation with the given general solution

$$y(x) = A\cos 2x + B\sin 2x + C\cosh 2x + D\sinh 2x$$

roots $\pm 2i$

$$= C\,\frac{e^{2x}+e^{-2x}}{2} + D\,\frac{e^{2x}-e^{-2x}}{2}$$

$$= \frac{C+D}{2}e^{2x} + \frac{C-D}{2}e^{-2x}$$

root 2 root -2

Recall

$$\cosh 2x = \frac{e^{2x}+e^{-2x}}{2}$$

$$\sinh 2x = \frac{e^{2x}-e^{-2x}}{2}$$

$P(D)\,y = 0$

?

Operator $(D^2+2^2)\begin{matrix}\cos 2x\\ \sin 2x\end{matrix} = 0$

Operator $(D-2)\,e^{2x} = 0$ Operator $(D+2)\,e^{-2x} = 0$

$\Rightarrow$ Operator $P(D) = (D^2+2^2)(D-2)(D+2)$ annihilates all functions: $\cos 2x$, $\sin 2x$, e^{2x} and e^{-2x} (i.e. $\cosh 2x$ and $\sinh 2x$)

$\Rightarrow$ DE is $P(D)y=0$ or $(D^2+2^2)\underbrace{(D-2)(D+2)}_{D^2-2^2}y=0$

$\Rightarrow (D^2+2^2)(D^2-2^2)y=0$

$\Rightarrow (D^2-4^2)y=0$ or $(D^2-16)y=0$ or $\boxed{y^{iv} - 16y = 0}$

3.5 Nonhomogeneous Equations. Method of Undetermined Coefficients

Ex How would we solve the following IVP:

$$y'' + y = 15e^{2x}, \quad y(0) = 7, \quad y'(0) = 1.$$

We have to find a function $y(x)$ that when acted by operator D^2+1, we get $15e^{2x}$, i.e.

$$(D^2+1)y = 15e^{2x}$$

Let's try $y(x) = e^{-x}$.

$(D^2+1)e^{-x} = e^{-x} + e^{-x} = 2e^{-x} \neq 15e^{2x}$ N.G.

$y = \cos x$: $(D^2+1)\cos x = -\cos x + \cos x = 0$ N.G.

$y = e^{2x}$: $(D^2+1)e^{2x} = 4e^{2x} + e^{2x} = 5e^{2x}$ N.G.

$y = 3e^{2x}$: $(D^2+1)(3e^{2x}) = 3\cdot 4e^{2x} + 3e^{2x} = 15e^{2x}$ ✓

Q Is $y(x) = 3e^{2x}$ a solution of the above IVP?

No. It satisfies DE $y'' + y = 15e^{2x}$ but DOES NOT satisfy ICs. We would need a solution that has two arbitrary constants and then we will be able to satisfy ICs.

What about

$$y(x) = 3e^{2x} + C_1 g_1(x) + C_2 g_2(x)$$

We need:

$$(D^2+1)\, y(x) = 15e^{2x}$$

$$(D^2+1)\left[3e^{2x} + C_1 g_1(x) + C_2 g_2(x)\right] =$$

$$= \underbrace{(D^2+1)\left[3e^{2x}\right]}_{15e^{2x}} + \underbrace{(D^2+1)\left[C_1 g_1(x) + C_2 g_2(x)\right]}_{0 \text{ should be}} \overset{\text{has to}}{\underset{\text{be}}{=}} 15e^{2x}$$

but $\underset{\pm i}{(D^2+1)}\left[C_1 \cos x + C_2 \sin x\right] = 0$

$\Rightarrow g_1(x) = \cos x,\; g_2(x) = \sin x$: solutions of the associated homogeneous DE

$y'' + y = 0$ or $(D^2+1)\, y = 0$

Hence,

$$\boxed{y(x) = 3e^{2x} + C_1 \cos x + C_2 \sin x}$$

the general solution of nonhomogeneous DE

$$y'' + y = 15e^{2x}$$

ICs : $y(0) = 7$, $y'(0) = 1$

$y(0) = 7 \Rightarrow 7 = 3\underbrace{e^0}_{1} + C_1 \underbrace{\cos 0}_{1} + C_2 \cancel{\sin 0}^{0} \Rightarrow 7 = 3 + C_1 \Rightarrow C_1 = 4$

$y'(x) = 6e^{2x} - C_1 \sin x + C_2 \cos x$

$y'(0) = 1 \Rightarrow 1 = 6 + C_2 \Rightarrow C_2 = -5$

$\therefore \boxed{y(x) = 3e^{2x} + 4\cos x - 5\sin x}$ solution of IVP

<u>General problem</u>: how to solve

$$P(D)\,y = R(x) \,?$$

(a) We need "a" solution $y_p(x)$ free of arbitrary constants such that $P(D)\,y_p = R(x)$. We call $y_p(x)$ the <u>particular solution</u>.

(b) We need the general solution of the associated homogeneous DE $P(D)y=0$, call it $y_c(x)$. $y_c(x)$ is the <u>complementary function</u>, and it has n arbitrary constants where n is the order of DE.

(c) Finally, the general solution of $P(D)y=R(x)$ is

$$y(x) = \underbrace{y_c(x)}_{\substack{\text{complementary function} \\ \text{(has } n \text{ arbitrary constants)}}} + \underbrace{y_p(x)}_{\substack{\text{particular solution} \\ \text{(has NO arbitrary constants)}}}$$

<u>Check</u> if $P(D)y \stackrel{?}{=} R(x)$

$$P(D)y(x) = P(D)\left[y_c(x)+y_p(x)\right] = \underbrace{P(D)y_c}_{=0} + \underbrace{P(D)y_p}_{R(x)} = R(x) \checkmark$$

$\blacksquare$

Ex Solve $y'' + y = 15e^{2x}$

Associated homogeneous DE is

$$y'' + y = 0$$

$$(D^2 + 1)y = 0$$

$$\pm i$$

$y_c(x) = C_1 \cos x + C_2 \sin x$

$y_p(x) = 3e^{2x}$ (found previously – more later)

$\therefore \quad y(x) = \underbrace{C_1 \cos x + C_2 \sin x}_{y_c} + \underbrace{3e^{2x}}_{y_p}$: the general solution

Ex Solve $(D^2 + 1)y = 4e^x$

$\pm i$

$y_p = 2e^x$

$y_c = C_1 \cos x + C_2 \sin x$

$\Rightarrow \quad y(x) = \underbrace{C_1 \cos x + C_2 \sin x}_{y_c} + \underbrace{2e^x}_{y_p}$: the general solution

Ex Solve $(D^2 - 3D + 2)y = 8$

$y_p = 4$

$(D^2 - 3D + 2)y = 0$

$(D-1)(D-2)y = 0$

$1, 2$

$y_c(x) = C_1 e^x + C_2 e^{2x}$

$\therefore\ y(x) = \underbrace{C_1 e^x + C_2 e^{2x}}_{y_c} + \underbrace{4}_{y_p}$: general solution

Method of Undetermined Coefficients

$P(D)y = R(x) \qquad (1)$

$y(x) = y_c(x) + y_p(x)$: general solution of (1)

We need to learn how to find y_p.

Ex $y'' + y = 6e^x + 5e^{2x}$

$$\underbrace{(D^2+1)}_{P(D)} y = \underbrace{6e^x + 5e^{2x}}_{R(x)}$$

We guess our particular solution as

$y_g(x) = K_1 e^x + K_2 e^{2x}$: candidate for particular solution y_p

We substitute y_g into the nonhomogeneous DE

$y'' + y = 6e^x + 5e^{2x}$

to compute coefficients K_1, K_2, called undetermined coefficients.

Note We DO NOT use ICs to find undetermined coefficients.

$y'' + y = 6e^{x} + 5e^{2x}$

(1) $y_g = K_1 e^{x} + K_2 e^{2x}$

$y_g' = K_1 e^{x} + 2K_2 e^{2x}$

(1) $y_g'' = K_1 e^{x} + 4K_2 e^{2x}$

$(K_1 + K_1)e^{x} + (K_2 + 4K_2)e^{2x} = 6e^{x} + 5e^{2x}$

$2K_1 e^{x} + 5K_2 e^{2x} = 6e^{x} + 5e^{2x}$

$\therefore \quad 2K_1 = 6 \qquad 5K_2 = 5$

$\Rightarrow \boxed{K_1 = 3, \quad K_2 = 1}$

Note we can equate like coefficients because functions e^{x} and e^{2x} are linearly independent.

$\Rightarrow \boxed{y_p(x) = 3e^{x} + e^{2x}}$: particular solution

$y'' + y = 6e^x + 5e^{2x}$

Associated homogeneous DE is

$y'' + y = 0$

$(D^2+1)y = 0$

$\pm i$

$\Rightarrow \boxed{y_c(x) = C_1 \cos x + C_2 \sin x}$ complementary function

$\boxed{y_p(x) = 3e^x + e^{2x}}$

$\therefore \quad y(x) = \underbrace{C_1 \cos x + C_2 \sin x}_{y_c} + \underbrace{3e^x + e^{2x}}_{y_p}$: the general solution of $y'' + y = 6e^x + 5e^{2x}$

__Note__: if ICs are given, <u>only now</u> we can use them to find C_1, C_2.

3.5 Method of Undetermined Coefficients

Consider

$$P(D)y = R(x) \tag{3.1}$$

The general solution is $y = y_c + y_p$ where y_c is the complementary function (with arbitrary constants) and $P(D)y_c = 0$. y_p is the particular solution (with no arbitrary constants) and $P(D)y_p = R(x)$. Suppose that there is an operator (with constant coefficients) $A(D)$ called an *annihilator* such that $A(D)R(x) = 0$. If we operate on both sides of (3.1) with $A(D)$ we get a higher order equation

$$A(D)P(D)y = A(D)R(x) = 0$$

Consider this new higher order equation

$$A(D)P(D)y = 0 \tag{3.2}$$

To find the general solution of (3.2) we need the roots of the polynomial $P(D)A(D)$; they are $r_1, r_2, \ldots, r_j$, $q_1, q_2, \ldots, q_k$, where $r_1, r_2, \ldots, r_j$ are roots of $P(D)$ and $q_1, q_2, \ldots, q_k$ are the roots of $A(D)$. Thus the general solution of (3.2) is

$$y = y_c + y_q$$

where y_c is generated by the roots of $P(D)$ and y_q is generated by the roots of $A(D)$.

Note 1. $r_1, r_2, \ldots, r_j$, $q_1, q_2, \ldots, q_k$ are roots of a (single) polynomial $P(D)A(D)$, thus make sure that if one of the q's is a repeated root to treat it properly.

Note 2. The general solution of (3.1) is also "a" solution of (3.2)

$$A(D)P(D)[y_c + y_p] = A(D)R(x) = 0$$

Thus, since $y_c + y_p$ is "a" solution of (3.2) and $y_c + y_q$ is "the general solution" of (3.2), $y_c + y_p$ must be contained in $y_c + y_q$, i.e.

$$(y_c + y_p) \subset (y_c + y_q) \quad \text{or} \quad y_p \subset y_q$$

We call y_q the "candidate" for the particular solutions and use the method of undetermined coefficients to evaluate the constants in y_q and thus find y_p.