Lecture 26

## Method of Undetermined Coefficients (Cont'd)

**Q** How to find $y_q$, the candidate for the particular solution $y_p$ for DE $P(D)y = R(x)$?

**Q** For which functions $R(x)$ we can apply the method of undetermined coefficients?

We can apply this method to DEs w/ constant coefficients for which $R(x)$ can be annihilated by some operator $A(D)$ with constant coefficients, called ANNIHILATOR.

**Ex** Find the candidate $y_q$ for particular solution $y_p$

$$\underbrace{y'' - 3y' + 2y}_{P(D)y} = \underbrace{8\cos 2x + 6e^{4x}}_{R(x)}$$

$\pm 2i$      $4$

$P(D) = D^2 - 3D + 2$

$A(D) = (D^2 + 2^2)(D - 4)$

$(D-a)e^{ax} = 0$

$$(D^2 + b^2)\begin{pmatrix}\cos 2x \\ \sin 2x\end{pmatrix} = 0$$

A higher order DE is

$$P(D)\,A(D)\,y = 0$$

$$\left\{ A(D) \mid P(D)y = R(x) \quad A(D)R(x) = 0 \quad A(D)\underbrace{P(D)y}_{"R(x)} = A(D)R(x) = 0 \right.$$

$$\underbrace{(D^2 - 3D + 2)}_{P(D)}\underbrace{\left[(D^2+2^2)(D-4)\right]}_{A(D)} y = 0$$

$$(D-1)(D-2)\left[(D^2+2^2)(D-4)\right] y = 0$$

$$1,\ 2;\quad \pm 2i,\ 4$$

$$y(x) = \underbrace{C_1 e^{x} + C_2 e^{2x}}_{y_c} + \underbrace{K_1 \cos 2x + K_2 \sin 2x + K_3 e^{4x}}_{y_q}$$

$$(D-1)^2 e^{x} \cdot x^{k} = 0 \qquad k = 0, 1$$

Ex $$\underbrace{(D^2+1)(D-1)(D+4)}_{P(D)} y = 10\cos 4x + 6xe^{x} - 12e^{-4x}$$

$$\pm 4i \qquad 1, 1 \qquad -4$$

$$A(D) = (D^2+4^2)(D-1)^2(D+4)$$

a higher order DE is

$P(D)A(D)y = 0$

$(D^2+1)(D-1)(D+4)\left[(D^2+4^2)(D-1)^2(D+4)\right]y = 0$

$\pm i,\ 1,\ -4;\qquad \pm 4i,\ 1, 1,\ -4$

$y_c$

$y(x) = C_1\cos x + C_2\sin x + C_3 e^x + C_4 e^{-4x} + K_1\cos 4x + K_2\sin 4x + K_5 x e^x + K_6 x^2 e^x + K_7 x e^{-4x}$

$y_g$

Ex $\underbrace{D^3(D-1)^2(D^2+1)}_{P(D)} y = 4x - 7x^2e^x + 9x^2e^{-x} + 3\cos x$

$0,0 \qquad 1,1,1 \qquad -1,-1,-1 \qquad \pm i$

$(D-0)\,\underbrace{e^{0x}}_{=1} = 0$

$A(D) = D^2(D-1)^3(D+1)^3(D^2+1)$

Higher order DE is

$$P(D)\,A(D)\,y = 0$$

$$D^3 (D-1)^2 (D^2+1)\left[ D^2 (D-1)^3 (D+1)^3 (D^2+1)\right] y = 0$$

$0,0,0,\ 1,1,\ \pm i;\quad 0,0,\ 1,1,1,\ -1,-1,-1,\ \pm i$

$$y(x) = \underbrace{C_1 + C_2 x + C_2 x^2 + C_3 e^x + C_4 x e^x + C_5 \cos x + C_6 \sin x}_{y_c} +$$

$$\underbrace{+ K_1 x^3 + K_2 x^4 + K_3 x^2 e^x + K_4 x^3 e^x + K_5 x^4 e^x + K_6 e^{-x} + K_7 x e^{-x} + K_8 x^2 e^{-x} + K_9 x \cos x + K_{10} x \sin x}_{y_g}$$

Q What do we do with $y_g$, the candidate for particular solution $y_p$? We substitute $y_g$ into the given <u>nonhomogeneous</u> DE $P(D)y = R(x)$ to find constants $K_i$'s to obtain particular solution $y_p$.

Ex Solve

$$\underbrace{y'' - 3y' + 2y}_{P(D)y} = xe^{2x} + \sin x, \qquad y(0) = 1.3, \quad y'(0) = 4.1$$

$xe^{2x} \to 2, 2$ $\qquad \sin x \to \pm i$

1. Identify RHS $R(x)$

$$A(D) = (D-2)^2(D^2+1)$$

2. Find annihilator $A(D)$

Higher order DE is

3. Form higher order DE

$$P(D)A(D)y = 0 \qquad (*)$$

$$(D^2 - 3D + 2)\left[(D-2)^2(D^2+1)\right]y = 0$$

$$(D-1)(D-2)\left[(D-2)^2(D^2+1)\right]y = 0$$

$1, 2; \qquad 2, 2, \pm i$

$$y(x) = \underbrace{C_1 e^x + C_2 e^{2x}}_{y_c} + \underbrace{K_1 xe^{2x} + K_2 x^2 e^{2x} + K_3 \cos x + K_4 \sin x}_{y_g}$$

— general solution of $(*)$

$y_g(x) = K_1 xe^{2x} + K_2 x^2 e^{2x} + K_3 \cos x + K_4 \sin x$ : candidate for particular solution

4. identify $y_g$

University of Idaho

To find $y_p$, we act on $y_g$ by operator $P(D)$ and equate the result to $R(x)$, i.e. we substitute $y_g$ into the nonhomogeneous DE $P(D)y = R(x)$ to find all $K_i$'s constants.

5

$$y'' - 3y' + 2y = xe^{2x} + \sin x$$

2 $\quad y_g = K_1 x e^{2x} + K_2 x^2 e^{2x} + K_3 \cos x + K_4 \sin x$

-3 $\quad y_g' = K_1 e^{2x} + (2K_1 + 2K_2) x e^{2x} + 2K_2 x^2 e^{2x} + K_4 \cos x - K_3 \sin x$

1 $\quad y_g'' = (2K_2 + 4K_1) e^{2x} + (8K_2 + 4K_1) x e^{2x} + 4K_2 x^2 e^{2x} - K_3 \cos x - K_4 \sin x$

$$\left[-3K_1 + (2K_2 + 4K_1)\right] e^{2x} + \left[2K_1 - 3(2K_1 + 2K_2) + (8K_2 + 4K_1)\right] x e^{2x} +$$

$$+ \left[\cancel{2K_2 - 3\cdot 2K_2 + 4K_2}\right] x^2 e^{2x} + \left[2K_3 - 3K_4 - K_3\right] \cos x +$$

0

$$+ \left[2K_4 - 3(-K_3) - K_4\right] \sin x = \underbrace{x e^{2x} + \sin x}_{R(x)}$$

$$[K_1+2K_2]e^{2x}+[2K_2]xe^{2x}+[K_3-3K_4]\cos x+[3K_3+K_4]\sin x =$$
$$= xe^{2x}+\sin x$$

6

To find undetermined coefficients $K_1, K_2, K_3, K_4$, we equate coefficients of $e^{2x}$, $xe^{2x}$, $\cos x$, $\sin x$ on the LHS to those on RHS. Q Why can we do this? We can do this because functions $e^{2x}$, $xe^{2x}$, $\cos x$, $\sin x$ are linearly independent.

$e^{2x}$ : $K_1+2K_2=0 \Rightarrow K_1=-2K_2=\boxed{-1=K_1}$

$xe^{2x}$ : $2K_2=1 \Rightarrow \boxed{K_2=\frac{1}{2}}$

$$\left.\begin{aligned} \cos x &: K_3-3K_4=0 \\ \sin x &: 3K_3+K_4=1 \end{aligned}\right\} \Rightarrow \boxed{K_3=0.3} \quad \boxed{K_4=0.1}$$

By equating like coefficients, we obtained a system of 4 equations for 4 unknowns $K_1, K_2, K_3, K_4$, which we can solve for $K_1, K_2, K_3, K_4$.

$$y_g = K_1 x e^{2x} + K_2 x^2 e^{2x} + K_3 \cos x + K_4 \sin x$$

$$K_1 = -1, \quad K_2 = \frac{1}{2}, \quad K_3 = 0.3, \quad K_4 = 0.1$$

$$\therefore \boxed{y_p = -x e^{2x} + \frac{1}{2} x^2 e^{2x} + 0.3 \cos x + 0.1 \sin x}$$

particular solution

$$y = y_c + y_p$$