Lecture 27

Method of Undetermined Coefficients (Cont'd)

Ex (Cont'd)

Solve $y'' - 3y' + 2y = xe^{2x} + \sin x$, $\quad y(0) = 1.3$, $y'(0) = 4.1$

Recall that $y_p(x)$ solves $P(D) = R(x)$ but may not satisfy ICs.

The general solution is

$y(x) = y_c(x) + y_p(x)$ ⑦ form general solution of $P(D)y = R(x)$

solution of $P(D)y = 0$

$$y(x) = \underbrace{C_1 e^x + C_2 e^{2x}}_{y_c} + \underbrace{\left[-xe^{2x} + \tfrac{1}{2}x^2 e^{2x} + 0.3\cos x + 0.1\sin x\right]}_{y_p}$$

the general solution of $y'' - 3y' + 2y = xe^{2x} + \sin x$

⑧ ONLY NOW WE CAN APPLY ICs TO FIND C_1, C_2.

$y(0)=1.3, \quad y'(0)=4.1$

$y'(x)=C_1e^x+2C_2e^{2x}-e^{2x}-xe^{2x}+x^2e^{2x}+0.1\cos x-0.3\sin x$

$$\left.\begin{aligned} y(0)=1.3 &\Rightarrow C_1+C_2+0.3=1.3 \\ y'(0)=4.1 &\Rightarrow C_1+2C_2-1+0.1=4.1 \end{aligned}\right\} \text{ system of two equations for unknowns } C_1, C_2$$

We find $\boxed{C_1=-3, \quad C_2=4}$

$\therefore \boxed{y(x)=-3e^x+4e^{2x}-xe^{2x}+\frac{1}{2}x^2e^{2x}+0.3\cos x+0.1\sin x}$

this is the solution of IVP $y''-3y'+2y=xe^{2x}+\sin x$, $y(0)=1.3$, $y'(0)=4.1$

Q What should we do when the RHS $R(x)$ cannot be annihilated by a linear differential operator w/ constant coefficients, i.e. there is no A(D). Or what if LHS is a linear DE w/ variable coefficients?

A We use a more general method to find y_p. The method is called VARIATION OF PARAMETERS. This method can also be applied to DEs w/ variable coefficients.

PROBLEM Solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = R(x), \quad a_2(x) \neq 0 \qquad (1)$$

where

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

ie. $y_1(x), y_2(x)$ are solutions of the associated homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

ie.

$$a_2 y_1'' + a_1 y_1' + a_0 y_1 = 0 \qquad (2)$$

$$a_2 y_2'' + a_1 y_2' + a_0 y_2 = 0 \qquad (3)$$

We assume (the method is due to Lagrange) that $y_p(x)$ has the same form as $y_c(x)$:

$$y_p(x) = A_1(x) y_1(x) + A_2(x) y_2(x)$$

$y_1(x), y_2(x)$: known functions

$A_1(x), A_2(x)$: unknow functions

Substituting y_p into the nonhomogeneous DE (1), we get

(a0) $y_p = A_1 y_1 + A_2 y_2$

(a1) $y_p' = A_1 y_1' + A_2 y_2' + [A_1' y_1 + A_2' y_2]$

(a2) $y_p'' = \underbrace{A_1 y_1'' + A_2 y_2''}_{0} + \{A_1' y_1' + A_2' y_2'\} + [A_1' y_1 + A_2' y_2]'$

$a_1[A_1' y_1 + A_2' y_2] + a_2\{A_1' y_1' + A_2' y_2'\} + a_2[A_1' y_1 + A_2' y_2]' = R(x)$

$a_1[\ldots] + a_2\{\ldots\} + a_2[\ldots]' = R(x)$

One equation for two unknowns $[\ldots]$ and $\{\ldots\}$. This implies that one variable should be a free parameter.

Let $[\ldots] = s = 0$

free parameter

$\Rightarrow a_2\{\ldots\} = R(x)$ or $a_2\{A_1' y_1' + A_2' y_2'\} = R(x) \;\Big|\; \frac{1}{a_2}$ $a_2(x) \neq 0$

$[\ldots] = 0$ $\Rightarrow$ $A_1' y_1 + A_2' y_2 = 0$

Thus,

$$\boxed{\begin{aligned} A_1' y_1 + A_2' y_2 &= 0 \\ A_1' y_1' + A_2' y_2' &= \frac{R(x)}{a_2(x)} \end{aligned}}$$

Two equations for two unknowns A_1', A_2'. In the matrix form

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{R(x)}{a_2(x)} \end{pmatrix}$$

This system has a unique solution $\begin{pmatrix} A_1' \\ A_2' \end{pmatrix}$ because determinant of the coefficient matrix

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \overline{W}(y_1, y_2) \neq 0$$ since y_1, y_2 are linearly independent

Solve for A_1', A_2'. Integrate and put back into y_p.

<u>Method of Variation of Parameters</u>

for $a_2(x)y'' + a_1(x)y' + a_0(x)y = R(x)$

① Solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

$\boxed{y_c(x) = C_1 y_1(x) + C_2 y_2(x)}$: <u>complementary function</u>

② Assume

$\boxed{y_p(x) = A_1(x)\, y_1(x) + A_2(x) y_2(x)}$

To find $A_1(x)$, $A_2(x)$, we solve for A_1', A_2' the following system

$$\boxed{\begin{aligned} A_1' y_1 + A_2' y_2 &= 0 \\ A_1' y_1' + A_2' y_2' &= \frac{R(x)}{a_2(x)} \end{aligned}}$$

(3) Once A_1', A_2' are known, integrate them to find A_1, A_2, and then

$$y_p(x) = A_1(x)y_1(x) + A_2(x)y_2(x).$$

(4) The general solution is

$$y(x) = \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{y_c} + \underbrace{A_1(x)y_1(x) + A_2(x)y_2(x)}_{y_p}$$

<u>NOTE</u> Whenever is possible to use the method of undetermined coefficients to find y_p, <u>do so</u>. Only in the cases when the method of undetermined coefficients is not applicable, use the method of variation of parameters.

<u>Ex</u> Solve $y'' - 3y' + 2y = \sin(e^{-x})$

$R(x) = \sin(e^{-x})$ cannot be annihilated by some $A(D)$

$\Rightarrow$ use method of variation of parameters

$y'' - 3y' + 2y = 0$

$(D^2 - 3D + 2)y = 0$

$(D-1)(D-2)y = 0$

$1, 2$ $\qquad y_1 = e^x, \quad y_2 = e^{2x}$

$y_c(x) = C_1 e^x + C_2 e^{2x} = C_1 y_1(x) + C_2 y_2(x)$

Assume

$y_p(x) = A_1(x) e^x + A_2(x) e^{2x} = A_1(x) y_1(x) + A_2(x) y_2(x)$

To find A_1, A_2, we solve for A_1', A_2' the system

$$\begin{cases} A_1' y_1 + A_2' y_2 = 0 \\ A_1' y_1' + A_2' y_2' = \dfrac{R(x)}{a_2(x)} \end{cases}$$

$$\begin{cases} A_1' \cdot e^x + A_2' \cdot e^{2x} = 0 \\ A_1' \cdot e^x + A_2' \cdot 2e^{2x} = \dfrac{\sin(e^{-x})}{1} \end{cases}$$

$y'' - 3y' + 2y = \underbrace{\sin(e^{-x})}_{R(x)}$

$\underset{a_2}{1} \cdot y''$

$$\begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sin(e^{-x}) \end{pmatrix}$$