Lecture 28

Cramer's Rule

$$\Delta = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x \cdot 2e^{2x} - e^x \cdot e^{2x} = \underbrace{e^{3x}}_{\neq 0} = W(e^x, e^{2x}) \neq 0$$

$$\Delta_1 = \begin{vmatrix} 0 & e^{2x} \\ \sin(e^{-x}) & 2e^{2x} \end{vmatrix} = -e^{2x} \cdot \sin(e^{-x})$$

$$\Delta_2 = \begin{vmatrix} e^x & 0 \\ e^x & \sin(e^{-x}) \end{vmatrix} = e^x \cdot \sin(e^{-x})$$

$$A_1' = \frac{\Delta_1}{\Delta} = \frac{-e^{2x} \cdot \sin(e^{-x})}{e^{3x}} = -e^{-x} \cdot \sin(e^{-x})$$

$$A_2' = \frac{\Delta_2}{\Delta} = \frac{e^x \cdot \sin(e^{-x})}{e^{3x}} = e^{-2x} \cdot \sin(e^{-x})$$

$$\therefore \quad A_1(x) = \int A_1'(x)\,dx = \int -e^{-x} \cdot \sin(e^{-x})\,dx = \begin{vmatrix} u = e^{-x} \\ du = -e^{-x}\,dx \end{vmatrix} =$$

$$= \int \sin u \cdot du = -\cos u = -\cos(e^{-x})$$

$$A_2(x) = \int A_2'(x)\,dx = \int e^{-2x} \cdot \sin(e^{-x})\,dx \overset{\text{integration by parts}}{=} -\sin(e^{-x}) + e^{-x}\cos(e^{-x})$$

$$\therefore\ y_p(x) = A_1 \cdot y_1 + A_2 \cdot y_2 = \underline{\cancel{-\cos(e^{-x})}} \cdot e^{x} + [-\sin(e^{-x}) + \underline{\cancel{e^{-x} \cdot \cos(e^{-x})}}]e^{2x} =$$

$$= -e^{2x} \cdot \sin(e^{-x})$$

$$\Rightarrow y_p(x) = -e^{2x} \cdot \sin(e^{-x})$$

Hence,

$$y(x) = \underbrace{C_1 e^{x} + C_2 \cdot e^{2x}}_{y_c} \underbrace{- e^{2x} \cdot \sin(e^{-x})}_{y_p} :$$ general solution of $y'' - 3y' + 2y = \sin(e^{-x})$

Ex Solve

$$\underbrace{x^2}_{a_2} \frac{d^2y}{dx^2} + \underbrace{x}_{a_1} \frac{dy}{dx} \underbrace{- y}_{a_0} = \underbrace{x}_{R(x)}, \qquad x > 0$$

The DE has variable coefficients $\Rightarrow$ method of undetermined coefficients is not applicable $\Rightarrow$ need to use the method of variation of parameters.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$: associated homogeneous DE

This is Euler equation or equidimensional equation:

each term has the form $x^n \frac{d^n y}{dx^n}$

power of x = order of derivative

Let $y(x) = x^p$

$$y' = p x^{p-1}, \qquad y'' = p(p-1)x^{p-2}$$

$$x^2 \cdot \underbrace{p(p-1)x^{p-2}}_{y''} + x \cdot \underbrace{p x^{p-1}}_{y'} - \underbrace{x^p}_{y} = 0$$

$$\underbrace{x^p}_{\neq 0}\left[p(p-1) + p - 1\right] = 0, \qquad x > 0$$

$\therefore \boxed{p(p-1)+p-1=0}$ characteristic equation

$(p-1)(p+1)=0 \Rightarrow p^2-1=0 \Rightarrow p=\pm 1 \Rightarrow y(x)=x^{\pm 1}$

or $y_1(x)=x$, $y_2(x)=\frac{1}{x}$

$\therefore y_c(x)=C_1 y_1(x)+C_2 y_2(x)=C_1\cdot x+C_2\cdot\frac{1}{x}$: complementary function

Let $y_p(x)=A_1(x)\cdot x+A_2(x)\cdot\frac{1}{x}$

To find A_1, A_2, we solve for A_1', A_2' the system

$$\begin{cases} A_1' y_1 + A_2' y_2 = 0 \\ A_1' y_1' + A_2' y_2' = \dfrac{R(x)}{a_2(x)} \end{cases}$$

$$\begin{cases} A_1'\cdot x + A_2'\cdot\frac{1}{x} = 0 \\ A_1'\cdot 1 + A_2'\cdot\left(-\frac{1}{x^2}\right) = \dfrac{x}{x^2} \end{cases}$$

$$\begin{pmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{x} \end{pmatrix}$$ Solve for A_1', A_2'

We find $A_1' = \frac{1}{2x}$, $A_2' = -\frac{x}{2}$

Integrate A_1', A_2' to get A_1, A_2:

$$A_1(x) = \int A_1'(x)\,dx = \int \frac{1}{2x}\,dx = \frac{1}{2}\ln x, \quad x>0$$

$$A_2(x) = \int A_2'(x)\,dx = \int -\frac{x}{2}\,dx = -\frac{x^2}{4}$$

$$\therefore\ y_p(x) = A_1(x)y_1(x) + A_2(x)y_2(x) = \frac{1}{2}\ln x \cdot x + \left(-\frac{x^2}{4}\right)\cdot\frac{1}{x}$$

or $\boxed{y_p(x) = \frac{1}{2}x\ln x - \frac{x}{4}}$

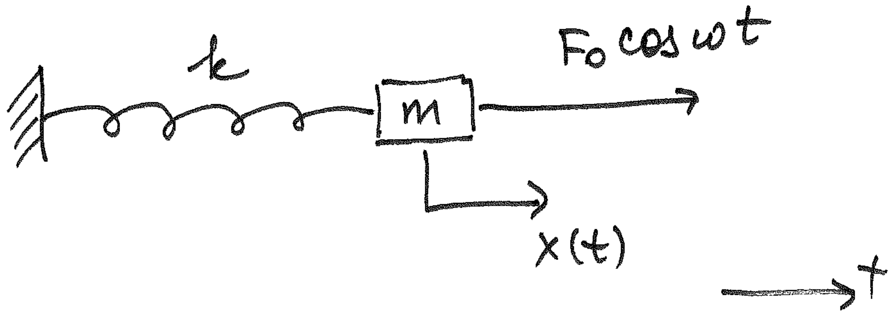
Then the general solution of $x^2y'' + xy' - y = x$ is

$$y(x) = \underbrace{c_1\cdot x + c_2\frac{1}{x}}_{y_c} + \underbrace{\frac{1}{2}x\ln x - \frac{x}{4}}_{y_p} = \underbrace{\left(c_1 - \frac{1}{4}\right)}_{=\tilde{c}_1}x + c_2\frac{1}{x} + \frac{1}{2}x\ln x$$

if C_1 is arbitrary $\Rightarrow$ $\tilde{C}_1 = C_1 - \frac{1}{4}$ is also arbitrary

$$\therefore \quad \boxed{y(x) = \tilde{C}_1 x + C_2 \frac{1}{x} + \frac{1}{2} x \ln x}$$

Section 3.6 DAMPED FORCED VIBRATIONS



$k>0$: spring const

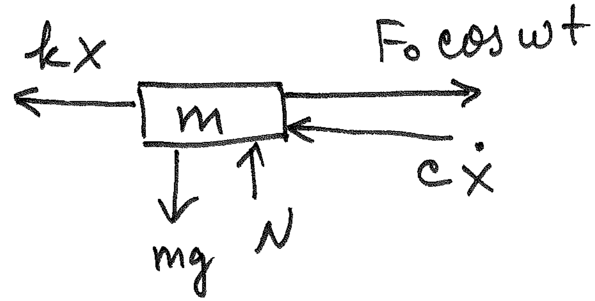
m: mass

$F_0 \cos \omega t$: applied external force

$x(t)$: displacement from equilibrium position

c: damping coefficient
$c>0$

FBD



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Newton's 2^{nd} law: $m\ddot{x} = F$

$$m\ddot{x} = -kx - c\dot{x} + F_0 \cos \omega t$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \qquad A(D) = D^2 + \omega^2$$

$$\pm i\omega$$

This is a 2^{nd} order linear DE w/ const coefficients, nonhomog.

Note that annihilator's roots: $\pm i\omega$ will NOT match (they may if $c=0$) roots of $P(D) = mD^2 + cD + k$

$\therefore$

(k) $x_g = K_1 \cos \omega t + K_2 \sin \omega t$

(c) $\dot{x}_g = \omega K_2 \cos \omega t - \omega K_1 \sin \omega t$

(m) $\ddot{x}_g = -\omega^2 K_1 \cos \omega t - \omega^2 K_2 \sin \omega t$

$$\cos \omega t \left(k K_1 + c\omega K_2 - m\omega^2 K_1\right) + \sin \omega t \left(k K_2 - c\omega K_1 - m\omega^2 K_2\right) =$$

$$= F_0 \cos \omega t$$