Lecture 29

#5 S3.5

$y'' + y' + y = \sin^2 x$

$y'' + y' + y = \frac{1}{2}(1 - \cos 2x)$

$\uparrow$ 0 $\quad$ $\uparrow$ $\pm 2i$

$A(D) = D(D^2+4)$

$\underbrace{(D^2 + D + 1)}_{P(D)} y = \frac{1}{2}(1 - \cos 2x)$

$D^2 + D + 1 = 0$

$\frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Higher order DE is

$(D^2+D+1)[D(D^2+4)]y = 0$

$-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; $\quad 0, \pm 2i$

$y(x) = \underbrace{C_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2}x + C_2 e^{-\frac{1}{2}x} \sin \frac{\sqrt{3}}{2}x}_{y_c} + \underbrace{K_1 + K_2 \cos 2x + K_3 \sin 2x}_{y_q}$

$\boxed{y_q = K_1 + K_2 \cos 2x + K_3 \sin 2x}$

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$$y^{(5)} + 2y^{(3)} + 2y'' = \underbrace{3x^2-1}_{0,0,0} \qquad A(D) = D^3$$

$$(D^5 + 2D^3 + 2D^2)y = 3x^2-1$$

$$\underbrace{D^2(D^3+2D+2)}_{P(D)}y = 3x^2-1$$

Higher order DE is

$$D^2(D^3+2D+2)[D^3]\, y = 0$$

using Maple

$0, 0, -0.77, 0.39 \pm i \cdot 1.56;\ 0, 0, 0$

$$y(x) = C_1 + C_2 x + C_3 e^{-0.77x} + C_4 e^{0.39x}\cos(1.56x) + C_5 e^{0.39x}\sin(1.56x) +$$

$$+ K_1 x^2 + K_2 x^3 + K_3 x^4$$

$$\boxed{y_q = K_1 x^2 + K_2 x^3 + K_3 x^4}$$

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S 3.5

$$y'' - 2y' + 2y = e^x \sin x \qquad A(D) = (D-1)^2 + 1$$

$$\uparrow$$
$$1 \pm i$$

$$(D^2 - 2D + 2)\, y = e^x \sin x$$

$$\left[(D-1)^2 + 1\right] y = e^x \sin x$$

Higher order DE is

$$\left[(D-1)^2 + 1\right]\left[(D-1)^2 + 1\right] y = 0$$

$$1 \pm i \, ; \qquad 1 \pm i$$

$$y(x) = \underbrace{c_1 e^x \cos x + c_2 e^x \sin x}_{y_c} + \underbrace{K_1 x e^x \cos x + K_2 x e^x \sin x}_{y_q}$$

$$\therefore \quad y_q = K_1 x e^x \cos x + K_2 x e^x \sin x$$

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S 3.5

$y'' + 4y = 3x\cos 2x \qquad A(D) = (D^2+4)^2$

$\uparrow$

$\pm 2i, \pm 2i$

$\underbrace{(D^2+4)}_{P(D)} y = 3x\cos 2x$

Higher order DE is

$(D^2+4)\left[(D^2+4)^2\right] y = 0$

$\pm 2i; \quad \pm 2i, \pm 2i$

$$y(x) = \underbrace{C_1\cos 2x + C_2\sin 2x}_{y_c} + \underbrace{K_1 x\cos 2x + K_2 x\sin 2x + K_3 x^2\cos 2x + K_4 x^2\sin 2x}_{y_g}$$

$$\therefore \boxed{y_g = K_1 x\cos 2x + K_2 x\sin 2x + K_3 x^2\cos 2x + K_4 x^2\sin 2x}$$

DAMPED FORCED VIBRATIONS (CONT'D)

$$\left.\begin{array}{ll} \cos\omega t: & (k-m\omega^2)K_1 + c\omega K_2 = F_0 \\ \sin\omega t: & -c\omega K_1 + (k-m\omega^2)K_2 = 0 \end{array}\right\} \text{system for } K_1, K_2$$

Solving this system, we find

$$K_1 = \frac{F_0 (k-\omega^2 m)}{(k-\omega^2 m)^2 + c^2\omega^2} \qquad K_2 = \frac{F_0 \cdot (c\omega)}{(k-\omega^2 m)^2 + c^2\omega^2}$$

$$\therefore \; x_p = \frac{F_0}{(k-\omega^2 m)^2 + c^2\omega^2}\left[(k-\omega^2 m)\cos\omega t + c\omega \cdot \sin\omega t\right]$$

Aside

$$C_1\cos\omega t + C_2\sin\omega t = \sqrt{C_1^2+C_2^2}\left(\frac{C_1}{\sqrt{\cdots}}\cos\omega t + \frac{C_2}{\sqrt{\cdots}}\sin\omega t\right) = \sqrt{C_1^2+C_2^2}\cos(\omega t - \alpha)$$

x_p can be written as

$$x_p = \frac{F_0}{\underbrace{\sqrt{(k-\omega^2 m)^2 + c^2\omega^2}}_{>0}}\cos(\omega t - \alpha)$$

where $\tan\alpha = \dfrac{\overbrace{c\omega}^{>0}}{\underbrace{k-\omega^2 m}_{<0,\,0,\,>0}}$ $\Rightarrow$ α is in the upper half plane, i.e. $0 \le \alpha \le \pi$ (since $c\omega > 0$)

General solution is

$$x(t) = C_1 e^{-p_1 t} + C_2 e^{-p_2 t} + \frac{F_0}{\sqrt{(k-\omega^2 m)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt} + \frac{F_0}{\sqrt{(k-\omega^2 m)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

$$x(t) = C_1 e^{-pt} \cos rt + C_2 e^{-pt} \sin rt + \frac{F_0}{\sqrt{(k-\omega^2 m)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

depending whether the system is overdamped, critically damped or underdamped. In all three cases, the first two terms approach 0 as $t \to \infty$, and the third term remains.

The first two terms are called TRANSIENT SOLUTION.

The last term is called STEADY STATE SOLUTION, $x_{ss}(t)$.

The steady state solution is

$$x_{ss}(t) = \underbrace{\frac{F_0}{\sqrt{(k-\omega^2 m)^2 + c^2\omega^2}}}_{\text{amplitude of the response}} \cos(\omega t - \alpha) \qquad \tan\alpha = \frac{c\omega}{k-\omega^2 m}$$

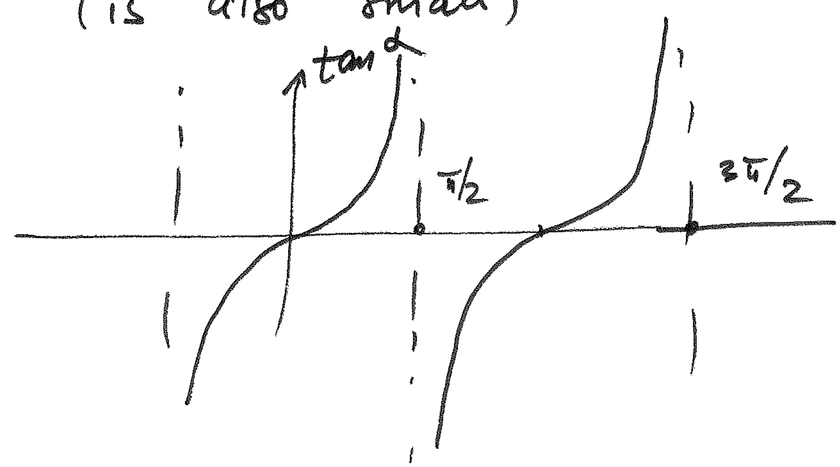
Note that both amplitude and phase α of the response function $x_{ss}(t)$ depend on the frequency of applied/driving force.

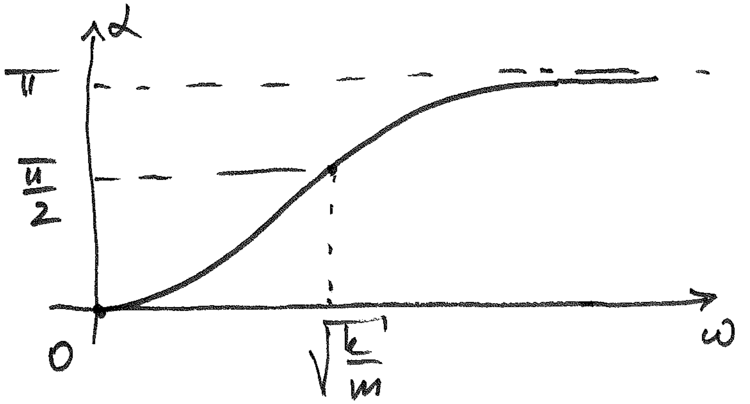
$$\tan\alpha = \frac{c\omega}{k-\omega^2 m}$$

if $\omega \ll 1$ (very small) $\Rightarrow$ $\alpha \ll 1$ (is also small)

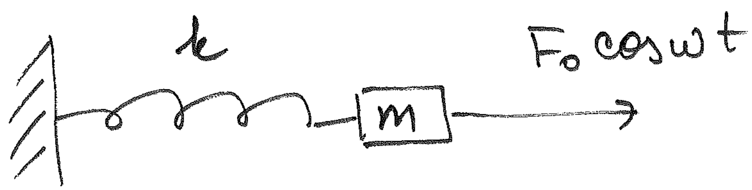
if $\omega = \sqrt{\frac{k}{m}}$ $\Rightarrow$ $\alpha = \frac{\pi}{2}$

if $\omega > \sqrt{\frac{k}{m}}$ $\Rightarrow$ $\frac{\pi}{2} < \alpha < \pi$





As $\omega \nearrow$, α also increases and varies between 0 and π



F_0 : amplitude of applied force

The amplitude of the response (i.e. $x_{ss}(t)$ steady state solution) is

$$B = \frac{F_0}{\sqrt{(k-\omega^2 m)^2 + c^2\omega^2}}$$

Possible graphs of B are

Note: B depends also on $F_0, k, m, c \Rightarrow$ different graphs

Q At which frequency should we drive the system to get maximum response, say B_{max}? This is called practical resonance.

B achieves its maximum value when

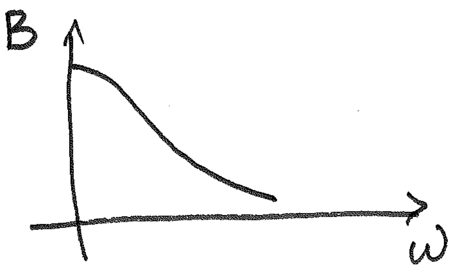
$$F(\omega) = (k-\omega^2 m)^2 + c^2\omega^2$$

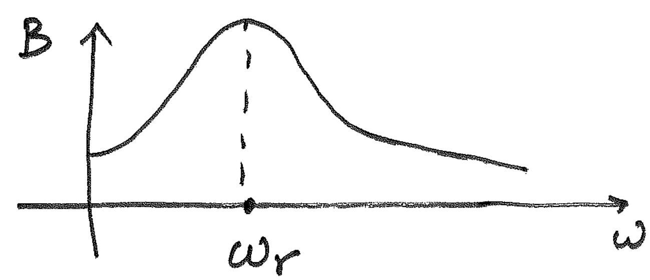
achieves its minimum.

$$F'(\omega) = 2(k-\omega^2 m)\cdot 2\omega m + 2c^2\omega = 2\omega\left[c^2 - 2(k-m\omega^2)m\right] = 0$$

$\omega = 0$ $\qquad \pm\sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$

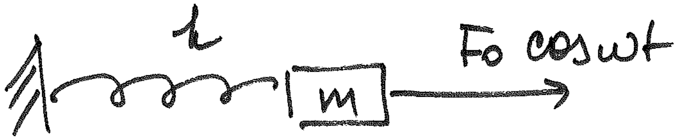
$$\omega_r = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} \quad : \text{resonating frequency}$$





$\omega = 0$ if $\frac{k}{m} - \frac{c^2}{2m^2} < 0$

UNDAMPED FORCED VIBRATIONS



$c = 0$: no damping

Then DE is

$$m\ddot{x} + kx = F_0 \cos\omega t$$

$$(mD^2 + k)x = F_0 \cos\omega t$$

$\pm i\omega_0$ $\qquad$ $\pm i\omega$