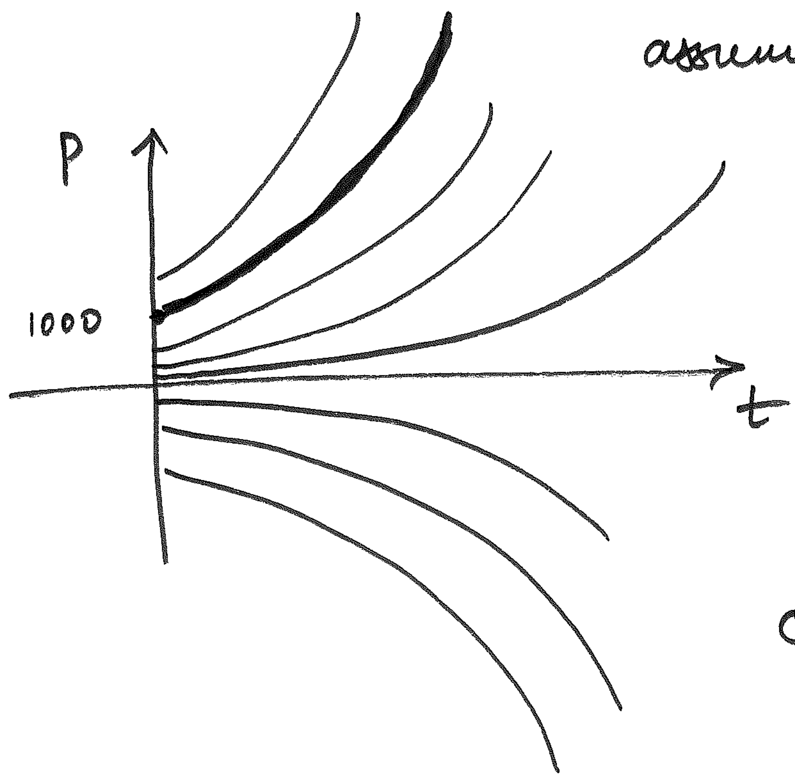
Lecture 2

Ex $\quad \frac{dP}{dt} = kP$

$P(t) = Ce^{kt}$, C: arbitrary const

assume $k > 0$



$C > 0$

$C = 0$

$C < 0$

IC

$P(0) = 1000$: <u>initial condition</u>

IC determines the unique solution out of infinitely many solutions defined by an arbitrary const C.

Def The order of a DE is the order of the highest derivative in DE.

Ex $y' = e^t$: 1st order ODE

$\frac{d^2x}{dt^2} + 9x = 0$: 2nd order DE

$y'' + 3y^3 = 2x$ $\qquad y = y(x)$

$y'' = \frac{d^2y}{dx^2}$ $\qquad y^3 = y \cdot y \cdot y$

2nd order DE

$y^{(3)} = y''' = \frac{d^3y}{dx^3}$

$y^{(4)} + y^2 x + y = \sin x$: 4th order DE

$y^{(4)} = \frac{d^4y}{dx^4}$ $\qquad y^2 = y \cdot y$

In general, n^{th} order DE for $y = y(x)$ can be written as

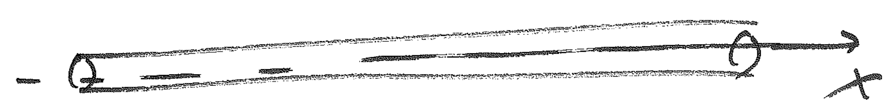
$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \qquad (1)$$

a continuous function $u = u(x)$ is a solution of (1) on some interval I if

$$F(x, u(x), u'(x), \dots, u^{(n)}(x)) \equiv 0 \quad \text{for all } x \in I$$

If we have a DE for function $u(x)$ of one variable x, we say we have an <u>ordinary differential equation</u> (ODE)

Consider $u = u(t, x)$: temperature in 1D thin uniform rod



One can show (Math 480):

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

K: thermal diffusivity

heat equation

partial derivatives

Eq$^{(n)}$ (2) is an example of a <u>partial differential equation</u>. Studied in Math 480 (PDEs).

This is an ODE class.

We will start from 1st order ODEs.

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) & y = y(x) \\ y(x_0) = y_0 & : \text{ initial condition} \end{cases}$$

DE + IC form <u>initial value problem</u> (IVP)

To solve IVP, one needs to find a function $y(x)$ that satisfies both DE & IC.

1.2 <u>Integrals as General and Particular solutions</u>

Consider $\dfrac{dy}{dx} = \underbrace{f(x)}_{\text{no } y}$

we want to find $y(x)$.

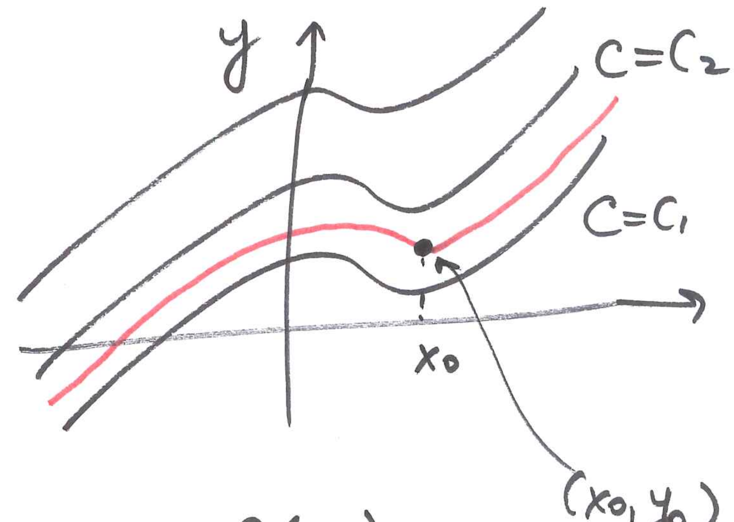
$$y(x) = \int \frac{dy}{dx}\,dx = \int f(x)\,dx + C$$

or $\boxed{y(x) = \int f(x)\,dx + C}$ a general solution of $\frac{dy}{dx} = f(x)$

It is one-parameter family of solutions.

Let $G(x)$ be an antiderivative of f, i.e. $G'(x) = f(x)$

Then $\boxed{y(x) = G(x) + C}$ general solution



IC: $y(x_0) = y_0$

at $x = x_0$: $\underbrace{y(x_0)}_{= y_0} = G(x_0) + C$

$\Rightarrow C = y_0 - G(x_0)$

$$\boxed{y(x) = G(x) + [y_0 - G(x_0)]}$$

particular solution of IVP

$\frac{dy}{dx} = f(x)$ subject to

IC $y(x_0) = y_0$

Note The general solution describes all possible solutions of a DE.

Ex Solve IVP

$\frac{dy}{dx} = 4x - 5,\quad y(1) = 2$

$y(x) = \int (4x-5)\,dx = 2x^2 - 5x + C$: the general solution

IC $y(1) = 2 \Rightarrow \underbrace{y(1)}_{=2} = 2 \cdot 1^2 - 5 \cdot 1 + C \Rightarrow \cancel{2} = \cancel{2} - 5 + C$

$$\boxed{C = 5}$$

$\boxed{y(x) = 2x^2 - 5x + 5}$: the <u>particular solution</u>

<u>Ex</u> Consider

$$\frac{d^2y}{dx^2} = g(x) \qquad \text{2nd order DE}$$

no y

Integrate wrt x once.

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \underbrace{\int g(x)\,dx}_{G(x)} + C_1 = G(x) + C_1$$

Integrate again:

$$y(x) = \int \frac{dy}{dx}\,dx = \int \left[G(x) + C_1\right] dx = \underbrace{\int G(x)\,dx}_{\equiv H(x)} + C_1 x + C_2$$

$$\boxed{y(x) = H(x) + C_1 x + C_2}$$

a general solution of $\frac{d^2y}{dx^2} = g(x)$

Velocity and acceleration

$x(t)$: position of a particle

m: mass of the particle

$F(t)$: force acting on the particle along its line of motion

$v(t) = \frac{dx}{dt}$: velocity $\Rightarrow$ $x(t) = \int v(t)\,dt$ or

$$x(t) = \int_{t_0}^{t} v(\tau)\,d\tau + x_0, \quad x_0 = x(t_0)$$

integral w/ variable upper limit

$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$: acceleration

Newton's 2^{nd} law of motion:

$$ma = F \Rightarrow m\frac{d^2x}{dt^2} = F$$: 2^{nd} order DE for $x(t)$

(F given)

Divide both sides by m:

$$\left\{\begin{array}{l} \dfrac{d^2x}{dt^2} = \dfrac{F}{m} \\ x(0) = x_0 : \text{ initial position} \\ \dfrac{dx}{dt}(0) = v_0 : \text{ initial velocity} \end{array}\right.$$

Aside:

$m\frac{dv}{dt} = F$: 1^{st} order for $v(t)$

Let $\frac{F}{m} = a$: const

$$\frac{d^2x}{dt^2} = a$$

Integrate once.

$$\frac{dx}{dt} = \int a\,dt = at + C_1$$

IC: $\frac{dx}{dt}(0) = v_0 \Rightarrow \underbrace{\frac{dx}{dt}(t=0)}_{= v_0} = \cancelto{0}{a \cdot 0} + C_1 \Rightarrow \boxed{C_1 = v_0}$

$$\frac{dx}{dt} = v(t) = at + v_0$$

Integrate again.

$$x(t) = \int (at + v_0)\,dt = \frac{a}{2}t^2 + v_0 t + C_2$$

IC: $x(0) = x_0 \Rightarrow x_0 = \cancelto{0}{\frac{a}{2} \cdot 0^2} + \cancelto{0}{v_0 \cdot 0} + C_2 \Rightarrow \boxed{C_2 = x_0}$

$\therefore \boxed{x(t) = \frac{a}{2}t^2 + v_0 t + x_0}$ $\boxed{v(t) = at + v_0}$

Physical Units

	British	Metric
Force	lb	N
Mass	slug	kg
Distance	ft	m
Time	s	s
g	$32 \frac{ft}{s^2}$	$9.8 \frac{m}{s^2}$