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Lecture 2

Recall the problem

$$\frac{dP}{dt} = kP$$

with solution $P(t) = Ce^{kt}$, C is an arbitrary constant.

Condition $P(0) = 1000$ is called an <u>initial condition</u> (IC).

IC determines the unique solution out of infinitely many solutions defined by an arbitrary constant C.

IC specifies C

Def The <u>order of a DE</u> is the order of the highest derivative in DE.

Ex $y' = e^t$: 1^{st} order DE

$\frac{d^2x}{dt^2} + 9x = 0$: 2^{nd} order DE

$y'' + 3\underbrace{y^3}_{y\cdot y\cdot y} = 2x$: 2^{nd} order DE

$y'' = \frac{d^2y}{dx^2}$

$y^3 = y\cdot y\cdot y$

$y^{(3)} = \frac{d^3y}{dx^3} = y'''$

$y^{(4)} + y^2 x + y = \sin x$: 4^{th} order DE

$y^{(4)} = y^{IV} = \frac{d^4y}{dx^4}$

In general, n^{th} order DE for $y = y(x)$ can be written as

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \qquad (1)$$

A continuous function $u = u(x)$ is a solution of DE (1) on some interval $x \in I$ if

$$F(x, u, u', \dots, u^{(n)}) \equiv 0 \quad \text{for all } x \in I$$

If we have a DE for function $u(x)$ of one variable x, then we have an <u>ordinary differential equation</u> (ODE)

Consider $u = u(t,x)$: temperature of a long thin uniform rod

Then

(2) $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$ $k>0$: thermal diffusivity

partial derivatives

Eqⁿ (2) is a <u>partial differential equation</u> (PDE). Math 480 – PDEs.

This is an ODE class.

<u>Aside:</u>

$$z = x^2 \cdot \sin y$$

$$\frac{\partial z}{\partial x} = 2x \cdot \sin y$$

$$\frac{\partial z}{\partial y} = x^2 \cdot \cos y$$

$$z = \cos(x^2 y)$$

$$\frac{\partial z}{\partial x} = -\sin(x^2 \cdot y) \cdot \underbrace{2xy}_{\frac{\partial}{\partial x}(x^2 y)}$$

$$\frac{\partial z}{\partial y} = -\sin(x^2 \cdot y) \cdot \underbrace{x^2}_{\frac{\partial}{\partial y}(x^2 y)}$$

We will start from 1st order DEs.

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \quad : \quad \text{IC} \end{cases} \qquad y = y(x)$$

DE + IC form an <u>initial value problem</u> (IVP).

To solve IVP, we need to find a function $y = y(x)$ that satisfies both DE and IC.

1.2 Integrals as General and Particular Solutions

Consider

$$\frac{dy}{dx} = f(x) \qquad \text{1st order DE}$$

no y

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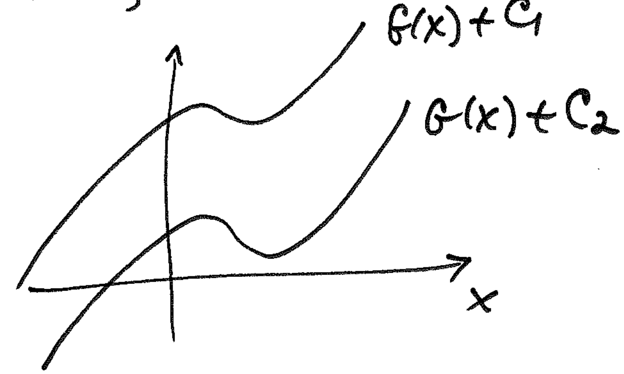
We want to find solution $y(x)$.

$$y(x) = \int \frac{dy}{dx}\,dx = \int f(x)\,dx + \underbrace{C}_{\text{arbitrary const}}$$

$$\boxed{y(x) = \int f(x)\,dx + C}$$ <u>a general solution</u> of DE $\frac{dy}{dx} = f(x)$

a general solution of a 1st order DE involves an arbitrary constant. It defines a one-parameter family of solutions.

Let $G(x)$ be an antiderivative of $f(x)$, i.e. $G'(x) = f(x)$.



$$\Rightarrow \boxed{y(x) = G(x) + C}$$

If we have IC $y(x_0) = y_0$.

$$\Rightarrow \text{at } x = x_0: \quad \underbrace{y(x_0)}_{= y_0} = \underbrace{G(x_0)}_{\text{can evaluate}} + C \Rightarrow \boxed{C = y_0 - G(x_0)}$$

$\Rightarrow \boxed{y(x) = G(x) + [y_0 - G(x_0)]}$ the particular solution of DE $y' = f(x)$ subject to IC $y(x_0) = y_0$.

<u>Note</u> The <u>general solution</u> describes all possible solutions of a DE.

<u>Ex</u> Solve IVP

$$\frac{dy}{dx} = 4x - 5, \quad y(1) = 2$$

$y(x) = \int (4x-5)\,dx = 2x^2 - 5x + C$: <u>the general solution</u>

at $x = 1$, $y(1) = 2$

$\underbrace{y(1)}_{=2} = 2\cdot 1^2 - 5\cdot 1 + C \Rightarrow 2 = 2 - 5 + C \Rightarrow C = 5$

$\therefore\ y(x) = 2x^2 - 5x + 5$: <u>the particular solution</u>

Consider

$$\frac{d^2y}{dx^2} = g(x) \quad : \qquad 2^{nd} \text{ order DE}$$

no y dependence

Integrate twice wrt x.

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \underbrace{\int g(x)\,dx}_{\equiv G(x)} + C_1 = G(x) + C_1$$

arbitrary constants

$$y(x) = \int \frac{dy}{dx}\,dx = \int [G(x) + C_1]\,dx = \underbrace{\int G(x)\,dx}_{\equiv H(x)} + C_1 x + C_2 =$$

$$= H(x) + C_1 x + C_2$$

$$\therefore \quad \boxed{y(x) = H(x) + C_1 x + C_2}$$

a general solution of 2^{nd} order DE $y'' = g(x)$

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Velocity and Acceleration

$x(t)$: particle position

m: mass of particle

$F(t)$: force acting on the particle along its line of motion

$\frac{dx}{dt} = v(t)$: velocity $\Rightarrow$ $x(t) = \int v(t)\,dt$

or

$$x(t) = \int_{t_0}^{t} v(\tau)\,d\tau + x(t_0)$$

integral w/ upper variable limit

$$\frac{d}{dt} x(t) = v(\tau)\Big|_{\tau=t} = v(t)$$

$\frac{d^2x}{dt^2} = \frac{dv}{dt} = a(t)$: acceleration

Newton's 2^{nd} law of motion: $ma = F$

$\Rightarrow m\frac{d^2x}{dt^2} = F$: 2^{nd} order DE for $x(t)$

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$$\left.\begin{array}{l} x(0)=x_0: \text{ initial position} \\ \frac{dx}{dt}(0) = v_0: \text{ initial velocity} \end{array}\right\} \text{ICs}$$

Assume for simplicity that $F = \text{const} \Rightarrow a = \text{const}$

$F = ma$

$$\frac{d^2x}{dt^2} = \frac{F}{m} \quad \text{or} \quad \frac{d^2x}{dt^2} = a$$

Integrate:

$$\frac{dx}{dt} = \int \frac{d^2x}{dt^2}\, dt = \int a\, dt = at + C_1$$

IC: $x(0) = x_0$ $\Rightarrow$ at $t=0$: $\underbrace{\frac{dx}{dt}(0)}_{= v_0} = a\overset{0}{\cancel{0}} + C_1 \Rightarrow \boxed{C_1 = v_0}$

$\frac{dx}{dt}(0) = v_0$

$$\boxed{\frac{dx}{dt} = at + v_0} \quad \text{or} \quad \boxed{v(t) = at + v_0}$$

Integrate again

$$x(t) = \int (at + v_0) = dt = \frac{at^2}{2} + v_0 t + C_2$$

IC: $x(0) = x_0 \Rightarrow$ $\underset{x_0}{\underset{\parallel}{x(0)}} = \cancel{\frac{a}{2} 0^2}^{\,0} + \cancel{v_0 \cdot 0}^{\,0} + C_2 \Rightarrow C_2 = x_0$

$$\therefore \boxed{x(t) = \frac{a}{2} t^2 + v_0 t + x_0}$$