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Lecture 30

$$\underbrace{(mD^2+k)}_{\pm i\omega_0}x = \underbrace{F_0\cos\omega t}_{\pm i\omega} \qquad A(D) = D^2+\omega^2$$

$\omega_0 = \sqrt{\frac{k}{m}}$: natural frequency

Higher order DE

$(mD^2+k)(D^2+\omega_0^2)x = 0$

$m(D^2+\omega_0^2)(D^2+\omega_0^2)x = 0$

$\pm i\omega_0$; $\pm i\omega_0$

Case I : $\omega = \omega_0$

$A(D) = D^2+\omega_0^2$

$$\underbrace{x(t)}_{\text{general solution}} = \underbrace{C_1\cos\omega_0 t + C_2\sin\omega_0 t}_{x_c} + \underbrace{K_1 t\cos\omega_0 t + K_2 t\sin\omega_0 t}_{x_g}$$

We can find that $K_1 = 0$, $K_2 = \frac{F_0}{2m\omega_0}$ $\Rightarrow$ $x_p(t) = \frac{F_0}{2m\omega_0} t\sin\omega_0 t$

$\therefore$ $\boxed{x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{2m\omega_0} t\sin\omega_0 t}$: general solution

$x(t)\to\infty$ as $t\to\infty$: pure resonance

We have reinforcement of the natural frequency ω_0 by

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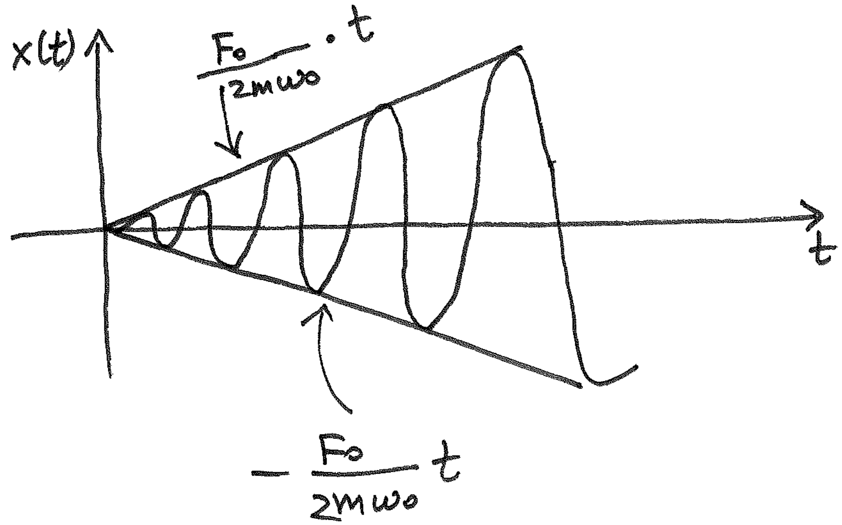
applying force with the same frequency.



Case II $\quad \omega \neq \omega_0$

$$x_g = K_1 \cos\omega t + K_2 \sin\omega t$$

We can find that

$$K_1 = \frac{F_0/m}{\omega_0^2 - \omega^2}, \quad K_2 = 0$$

$$\therefore \; x_p = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos\omega t$$

General solution is

$$x(t) = \underbrace{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t}_{x_c} + \underbrace{\frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t}_{x_p}$$

We have superposition of two oscillations w/ frequencies ω_0 and ω.

Beats

$\omega_0 \neq \omega$

Let $x(0) = \dot{x}(0) = 0 \Rightarrow C_1 = -\frac{F_0/m}{\omega_0^2 - \omega^2}, \quad C_2 = 0$

Then

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \left(\cos \omega_0 t - \cos \omega t \right)$$

Trig. identity: $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

$\alpha - \beta = \omega_0 t \qquad \alpha + \beta = \omega t$

$\Rightarrow \alpha = \frac{1}{2}(\omega_0 + \omega) t, \qquad \beta = \frac{1}{2}(\omega_0 - \omega) t$

then

$$\boxed{x(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\tfrac{1}{2}(\omega_0 - \omega)t \cdot \sin\tfrac{1}{2}(\omega_0 + \omega)t}$$

Suppose that $\omega \approx \omega_0 \Rightarrow \omega_0 + \omega >> \omega - \omega_0$

$\sin\frac{1}{2}(\omega_0 + \omega)t$: rapidly oscillating function

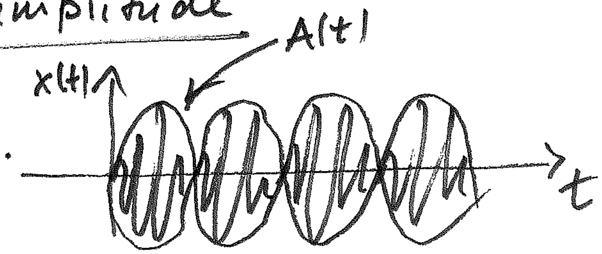
$\sin\frac{1}{2}(\omega_0 - \omega)t$: slowly oscillating function

$$x(t) = \left(\frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\tfrac{1}{2}(\omega_0 - \omega)t\right) \sin\tfrac{1}{2}(\omega_0 + \omega)t$$

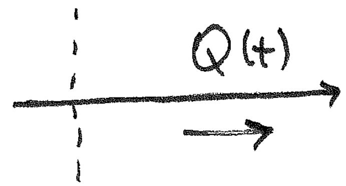
We can interpret $x(t)$ as rapidly oscillating function of frequency $\frac{1}{2}(\omega_0 + \omega)$ and slowly varying amplitude

$$A(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\tfrac{1}{2}(\omega_0 - \omega)t :$$ modulation / change of amplitude

This type of solution is called BEATS.



3.7 Electric Circuits



Consider a wire through which charges measured in coulombs flow.

⋮ denotes an imaginary plane that "cuts" the wire (or reference plane).

Let $Q(t)$ be charge (in coulombs). Define $\frac{dQ}{dt}$, the rate at which charges flow past our reference plane. Current $I(t) = \frac{dQ}{dt}$, measured in $\frac{\text{coulombs}}{\text{sec}}$ = amperes. Hence,

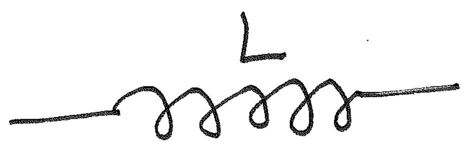
$$I(t) = \frac{dQ}{dt} \quad \text{or} \quad Q(t) = \int_{t_0}^{t} I(\tau)\,d\tau + Q(t_0)$$

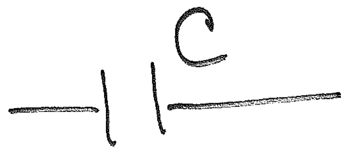
If $\frac{dQ}{dt} > 0$, then charges flow from left to right.

If $\frac{dQ}{dt} < 0$, — " — " — " — right to left.

why do charges flow at all? We assume that there is some potential (voltage) that makes charges flow from higher potential to lower potential. We measure voltage $E(t)$ drop in volts. Assume that we have three elements in the circuit: inductor, resistor and capacitor. We would like to have relation between $I(t)$, $Q(t)$ and $E(t)$.

Resistor  $V_R = RI$ Ohms

Inductor  $V_L = L\dfrac{dI}{dt}$ Henries

Capacitor  $V_C = \dfrac{1}{C} Q$ Farads

Consider a circuit that has these three elements.

<u>2^{nd} KIRCHHOFF's LAW</u>: Algebraic sum of voltage drops due to elements in a circuit equals applied

voltage $E(t)$.

$$\boxed{L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)} \qquad (1)$$

1st order DE but it is coupled since it depends on both $Q(t)$ and $I(t)$

Recall $\quad \frac{dQ}{dt} = I(t)\,, \quad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$

$$\Rightarrow \boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t),} \qquad (2)$$

2nd order DE for $Q(t)$

$Q(0) = Q_0$: initial charge

$I(0) = I_0 = \frac{dQ}{dt}(0)$: initial current