Lecture 31

Electric Circuits (Cont'd)

We derived eq^{n}

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t) \quad \Big| \ \frac{d}{dt}$$

Differentiate both sides of this eq^{n} wrt t

$$\boxed{L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)} \qquad (3)$$

2nd order DE for $I(t)$

since $\frac{dQ}{dt} = I(t)$.

Usually we will given initial charge $Q(0) = Q_0$ and initial current $I(0) = I_0$, but to solve eq^{n} (3) we would need $I(0) = I_0$ and $\frac{dI}{dt}(0)$ — ? How do find $\frac{dI}{dt}(0)$.

<u>a</u> We will use eq^{n} (1)

$$\boxed{L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)} \qquad (1)$$

and solve it for $\frac{dI}{dt}$ at $t=0$.

known, given as ICs

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{E(0) - R\,I(0) - \frac{1}{C}\,Q(0)}{L}$$

Consider again eq^{n} (1), and recall $\frac{dQ}{dt} = I(t)$ or

$$Q(t) = \int_{t_0}^{t} I(\tau)\,d\tau + Q(t_0)$$

Substitute $Q(t)$ into (1) to get

$$\boxed{L\frac{dI}{dt} + RI + \frac{1}{C}\Big[\int_{t_0}^{t} I(\tau)\,d\tau + \underbrace{Q(t_0)}_{\text{const}}\Big] = E(t)} \qquad (4)$$

Eq^{n} (4) is an <u>integro-differential equation</u> for $I(t)$.

<u>**Note**</u> If we need to find charge $Q(t)$, we can use eq^{n} (2):

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \qquad Q(0) = Q_0, \qquad \frac{dQ}{dt}(0) = I(0) = I_0$$

If we need to find current $I(t)$, we can

Method I

Use eq^{n} (2) to solve for $Q(t)$ and then $I(t) = \frac{dQ}{dt}$

Method II

Solve eq^{n} (3) for $I(t)$ directly and use $\frac{dI}{dt}$ from eq^{n} (1), i.e.

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = E'(t),$$

$I(0) = I_0$

$$\frac{dI}{dt}(0) = \frac{E(0) - R\overbrace{I(0)}^{I_0} - \frac{1}{C}\overbrace{Q(0)}^{Q_0}}{L} = \frac{E(0) - R I_0 - \frac{1}{C} Q_0}{L}$$

There is an excellent analogy with mass-spring systems:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m \to L, \quad c \to R, \quad k \to \frac{1}{C}$$

$m \longleftrightarrow L$ stores energy in magnetic field

$c \longleftrightarrow R$ dissipates energy

$k \longleftrightarrow \frac{1}{C}$ stores energy in electric field

$x(t) \longleftrightarrow Q(t)$

$\dot{x}(t) \longleftrightarrow I(t)$

$\frac{1}{2}m(\dot{x})^2$ kinetic energy $\longleftrightarrow$ $\frac{1}{2}LI^2$ magnetic energy

$\frac{1}{2}kx^2$ potential energy $\longleftrightarrow$ $\frac{1}{2C}Q^2$ electric energy

Similar to mass-spring systems, circuits can be overdamped, critically damped or oscillatory / underdamped.

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

$$\left(LD^2 + RD + \frac{1}{C}\right)Q = 0$$

$$\frac{-R \pm \sqrt{R^2 - 4 \cdot L \frac{1}{C}}}{2L}$$

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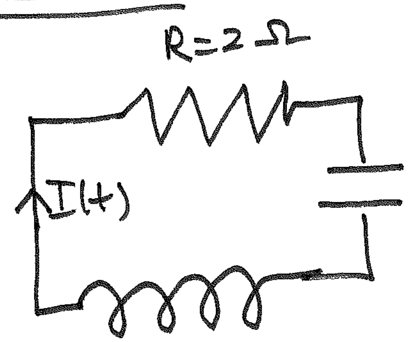
everything depends on the sign of discriminant $R^2 - 4L/C$

$R^2 > \frac{4L}{C}$: overdamped circuit

$R^2 = \frac{4L}{C}$: critically damped

$R^2 < \frac{4L}{C}$: oscillatory / underdamped

Problem Find $I(t)$ given that $I(0) = \frac{1}{10}$ and $Q(0) = -\frac{1}{65}$.



$C = \frac{1}{260}$ farads

$L = \frac{1}{10}$ henries

We will use eqn (3)

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0 \qquad (3)$$

$I(0) = \frac{1}{10}$, $\frac{dI}{dt}(0)$ — ? use 1st order coupled eq² for I and Q (eq² (1))

$$\frac{dI}{dt}(0) = \frac{\overset{0}{\cancel{E}(0)} - R\overset{\frac{1}{10}}{I(0)} - \frac{1}{C}\overset{-\frac{1}{65}}{Q(0)}}{L} = \frac{-2\cdot\frac{1}{10} - 260\left(-\frac{1}{65}\right)}{\frac{1}{10}} = 38$$

$$\frac{1}{10}\frac{d^2I}{dt^2} + 2\frac{dI}{dt} + 260\, I = 0, \quad I(0) = \frac{1}{10}, \quad \frac{dI}{dt}(0) = 38$$

We can use an operator approach, for example.

$$\left(\frac{1}{10}D^2 + 2D + 260\right) I = 0$$

$$\frac{-2 \pm \sqrt{2^2 - 4\cdot\frac{1}{10}\cdot 260}}{2\cdot\frac{1}{10}} = -10 \pm 50i$$

$$I(t) = C_1 e^{-10t}\cos 50t + C_2 e^{-10t}\sin 50t$$

To find C_1 and C_2, we use ICs $I(0) = \frac{1}{10}$, $\frac{dI}{dt}(0) = 38$

We can find $C_1 = 0.1$, $C_2 = 0.78$

since $I(0) = C_1$

$$\frac{dI}{dt} = C_1(-10)e^{-10t}\cos 50t - 50C_1e^{-10t}\sin 50t - 10C_2e^{-10t}\sin 50t +$$
$$+50C_2e^{-10t}\cos 50t$$

$$\frac{dI}{dt}(0) = -10\underset{0.1}{\underset{\parallel}{C_1}} + 50C_2 = 38 \Rightarrow \quad C_2 = 0.78$$

Chapter 7 Laplace Transforms

Ex Consider $\int_1^3 (s+t)^3\,dt$: s is a parameter

$$\int_1^3 (s+t)^3\,dt = \frac{(s+t)^4}{4}\Bigg|_{t=1}^{t=3} = \frac{(s+3)^4}{4} - \frac{(s+1)^4}{4} \text{ : function of } s$$

Note $\int_1^3 (s+t)^3\,dt = \int_1^3 (s+x)^3\,dx = \int_1^3 (s+p)^3\,dp$

Variable of integration in <u>definite</u> integrals is called a "dummy variable".

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Ex $\int_1^\infty \frac{1}{x^2+1}\,dx$: improper integral

$$\int_1^\infty \frac{1}{x^2+1}\,dx \overset{def}{=} \lim_{M\to\infty} \int_1^M \frac{1}{x^2+1}\,dx = \lim_{M\to\infty} \arctan x \Big|_{x=1}^{M} =$$

$$= \lim_{M\to\infty}\Big(\arctan M - \underbrace{\arctan 1}_{=\pi/4}\Big) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \ : \ \text{finite \#} \Rightarrow \int_1^\infty \frac{dx}{x^2+1} \ \underline{\text{converges}}$$