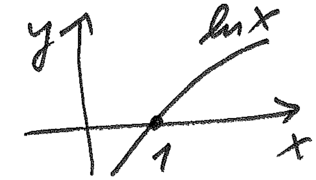
Lecture 32

<u>Ex</u> $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{M\to\infty} \int_1^M \frac{dx}{x} = \lim_{M\to\infty} \ln x \Big|_1^M = \lim_{M\to\infty} \ln M - \cancelto{0}{\ln 1} =$

$\displaystyle = \lim_{M\to\infty} \ln M = \infty \Rightarrow \int_1^\infty \frac{dx}{x}$ diverges



We learned about operator $D = \frac{d}{dx}$. For example, operator D^2+1 acts on a function $f(x)$ by differentiating it twice and adding this function:

$$(D^2+1)\, f(x) = \frac{d^2 f}{dx^2} + f(x)$$

$D^2+1: \; f \to f'' + f$: another function of x

This is an example of a differential operator. We could consider an integral operator

$$\int_a^b K(s,x)\, f(x)\, dx$$

where $K = K(s,x)$ is a kernel of this integral operator.

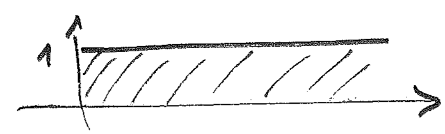
$\int_a^b K(s,x)\, f(x)\, dx$ is a function of s

if you give me a function $f(x)$, the result is some function $G(s)$ of variable s.

One of the extremely important integral operators is the one with $a=0$, $b=\infty$ and $K(s,x)=e^{-sx}$.

<u>Def</u> a <u>Laplace transform</u> of a function $f(x)$ is

$$\boxed{\mathcal{L}[f(x)] \overset{def}{=} \int_0^\infty e^{-sx} f(x)\, dx = F(s)}$$



provided that this integral exists.

<u>Ex</u> Find Laplace transform of e^{ax}.

$$\mathcal{L}\{e^{ax}\} \overset{def}{=} \int_0^\infty e^{-sx} e^{ax}\, dx = \int_0^\infty e^{(a-s)x}\, dx = \lim_{M\to\infty} \int_0^M e^{(a-s)x}\, dx =$$

$$= \lim_{M\to\infty} \frac{e^{(a-s)x}}{a-s} \Big|_{x=0}^{x=M} = \frac{1}{a-s}\Big(\underbrace{\cancelto{0}{\lim_{M\to\infty} e^{(a-s)M}}}_{\text{if } a-s<0} - \underbrace{e^{(a-s)\cdot 0}}_{=1} \Big) = \frac{1}{s-a}$$

$s>a$ for convergence

$$\boxed{\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}, \qquad s>a}$$

<u>Thm</u> Laplace transform is a linear operator, ie.

$$\mathcal{L}\{c_1 f_1(x) + c_2 f_2(x)\} = c_1 \mathcal{L}\{f_1(x)\} + c_2 \mathcal{L}\{f_2(x)\}$$

<u>Ex</u> What is $\mathcal{L}\{1\} = \mathcal{L}\{e^{0\cdot x}\} = \frac{1}{s-0} = \frac{1}{s}, \quad s>0$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}, \qquad s>0}$$

<u>Ex</u> What are $\mathcal{L}\{\cos kx\}$ and $\mathcal{L}\{\sin kx\}$?

$$e^{ikx} \underset{\text{formular}}{\overset{\text{Euler}}{=}} \cos kx + i \sin kx$$

$$\mathcal{L}\{e^{ikx}\} = \frac{1}{s-ik} = \frac{s+ik}{(s-ik)(s+ik)} = \frac{s+ik}{s^2+k^2} = \underline{\underline{\frac{s}{s^2+k^2}}} + i\,\underline{\underline{\frac{k}{s^2+k^2}}}$$

$a = ik$

$$\| \quad \mathcal{L}\{\cos kx + i \sin kx\} \underset{\text{a linear operator}}{\overset{\mathcal{L}\text{ is}}{=}} \underline{\underline{\mathcal{L}\{\cos kx\}}} + i\,\underline{\underline{\mathcal{L}\{\sin kx\}}}$$

$$\Rightarrow \quad \boxed{\mathcal{L}\{\cos kx\} = \frac{s}{s^2+k^2}} \qquad \boxed{\mathcal{L}\{\sin kx\} = \frac{k}{s^2+k^2}}$$

Now memorize

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}\{e^{ax}\} = \frac{1}{s-a} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax}$$

$$\mathcal{L}\{\cos kx\} = \frac{s}{s^2+k^2} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kx$$

$$\mathcal{L}\{\sin kx\} = \frac{k}{s^2+k^2} \qquad \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kx$$

$\mathcal{L}^{-1}$: inverse Laplace transform

$$\mathcal{L}\{f(x)\} = F(s) \quad \Rightarrow \quad \mathcal{L}^{-1}\{F(s)\} = f(x)$$

Problem Find $\mathcal{L}\{3e^{4t} - 6\cos 2t + 8\}$ - ? $=$

Laplace transform is a linear operator $\Rightarrow$

$$= 3\,\mathcal{L}\{e^{4t}\} - 6\,\mathcal{L}\{\cos 2t\} + 8\,\mathcal{L}\{1\} = 3\cdot\frac{1}{s-4} - 6\cdot\frac{s}{s^2+2^2} + 8\cdot\frac{1}{s}$$

$$= \frac{3}{s-4} - \frac{6s}{s^2+4} + \frac{8}{s}$$

Problem

$$\mathcal{L}\{5\sin\tfrac{1}{2}t - 3e^{-4t} + \cos 6\pi t - 4\} = 5\,\mathcal{L}\{\sin\tfrac{1}{2}t\} - 3\,\mathcal{L}\{e^{-4t}\} +$$

$$+ \mathcal{L}\{\cos 6\pi t\} - 4\,\mathcal{L}\{1\} = 5\cdot\frac{\frac{1}{2}}{s^2+(\frac{1}{2})^2} - 3\cdot\frac{1}{s-(-4)} + \frac{s}{s^2+(6\pi)^2} - 4\cdot\frac{1}{s} =$$

$$= \frac{5}{2}\,\frac{1}{s^2+\frac{1}{4}} - \frac{3}{s+4} + \frac{s}{s^2+36\pi^2} - \frac{4}{s}$$

Recall def of Laplace transform

$$\mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx = F(s)$$

Ex $\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}$, $\mathcal{L}\{\cos kx\} = \frac{s}{s^2+k^2}$, $\mathcal{L}\{\sin kx\} = \frac{k}{s^2+k^2}$

In the def of Laplace transform, s is a parameter. If instead of s we use another parameter, then Laplace transform will be a function of that other parameter.

$$\underbrace{\int_0^\infty e^{-st} \sin 2t \, dt}_{\mathcal{L}\{\sin 2t\}} = \frac{2}{s^2+2^2}$$

$$\int_0^\infty e^{-pt} \sin 2t \, dt = \frac{2}{p^2+2^2}$$

$$\int_0^\infty e^{-q^3 t} \sin 2t \, dt = \frac{2}{(q^3)^2+2^2} = \frac{2}{q^6+2^2}$$

$$\int_0^\infty \underbrace{e^{-(m+3)t}}_{e^{-mt} \cdot e^{-3t}} \sin 2t \, dt = \frac{2}{(m+3)^2+2^2}$$

$$\int_0^\infty e^{-mt} \cdot \underbrace{e^{-3t} \sin 2t}_{\mathcal{L}\{e^{-3t} \sin 2t\}} \, dt = \frac{2}{(m+3)^2+2^2}$$

Thm $\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}$ where $F(s) = \mathcal{L}\{f(t)\}$

Proof

$$\mathcal{L}\{e^{at} f(t)\} \overset{def}{=} \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a) \quad \blacksquare$$

Ex $\mathcal{L}\{e^{2t}\cos 8t\} = ?$ $\qquad \mathcal{L}\{\cos 8t\} = \frac{s}{s^2+8^2}$

$a=2 \Rightarrow s \to s-2$

$$\therefore \mathcal{L}\{e^{2t}\cos 8t\} = \frac{s-2}{(s-2)^2+8^2}$$

Ex $\mathcal{L}\{e^{-5t}\sin 3t\} - ?$ $\qquad \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+3^2}$

$a=-5 \qquad s \to s-(-5) = s+5$

$$\therefore \mathcal{L}\{e^{-5t}\sin 3t\} = \frac{3}{(s+5)^2+3^2}$$