Lecture 33

Review for Exam #2

- DEs w/ constant coefficients
- Operator approach
  operator identities will be provided (6)
- Nonhomogeneous equations:
  - method of undetermined coefficients
  - variation of parameters

- applications:
  - mechanical systems (mass-spring system w/ and w/o damping, resonance, practical resonance)
  - electric circuits

You may bring and use a half page of your notes

#61 S 3.5 Find a particular solution of

$x^2 y'' + x y' + y = \ln x$, $\quad y_c = C_1 \underbrace{\cos(\ln x)}_{y_1(x)} + C_2 \underbrace{\sin(\ln x)}_{y_2(x)}$

We use variation of parameters method.

Assume $y_p(x) = A_1(x) \cdot \cos(\ln x) + A_2(x) \sin(\ln x)$ $\qquad x > 0$

To find A_1, A_2, solve for A_1', A_2' the system

$$\begin{cases} A_1' y_1 + A_2' y_2 = 0 \\ A_1' y_1' + A_2' y_2' = \dfrac{R(x)}{a_2(x)} \end{cases}$$

Here $R(x) = \ln x$

$a_2(x) = x^2$

$a_2(x) y'' + a_1(x) y' + a_0(x) y = R(x)$

or

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{R(x)}{a_2(x)} \end{pmatrix}$$

$$\begin{pmatrix} \cos(\ln x) & \sin(\ln x) \\ -\sin(\ln x) \cdot \frac{1}{x} & \cos(\ln x) \cdot \frac{1}{x} \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{\ln x}{x^2} \end{pmatrix}$$

Use Cramer's rule to solve the system.

$$\Delta = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\sin(\ln x)\cdot\frac{1}{x} & \cos(\ln x)\cdot\frac{1}{x} \end{vmatrix} = \cos^2(\ln x)\cdot\frac{1}{x} + \sin^2(\ln x)\cdot\frac{1}{x} = \frac{1}{x} \neq 0 \quad x>0$$

$$\Delta = W(y_1, y_2) \neq 0$$

$$\Delta_1 = \begin{vmatrix} 0 & \sin(\ln x) \\ \frac{\ln x}{x^2} & \cos(\ln x)\cdot\frac{1}{x} \end{vmatrix} = -\frac{\ln x}{x^2}\sin(\ln x)$$

$$\Delta_2 = \begin{vmatrix} \cos(\ln x) & 0 \\ -\sin(\ln x)\cdot\frac{1}{x} & \frac{\ln x}{x^2} \end{vmatrix} = \frac{\ln x}{x^2}\cos(\ln x)$$

$$\therefore \quad A_1' = \frac{\Delta_1}{\Delta} = -\frac{\ln x\cdot\frac{1}{x^2}\sin(\ln x)}{\frac{1}{x}} = -\frac{\ln x}{x}\sin(\ln x)$$

$$A_2' = \frac{\Delta_2}{\Delta} = \frac{\ln x}{x}\cos(\ln x)$$

$$A_1(x) = \int A_1'(x)\,dx = -\int \frac{\ln x}{x}\sin(\ln x)\,dx = \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \right| = -\int u\cdot\sin u\,du \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$= \left| \begin{matrix} U = u & dV = \sin u\,du \\ dU = du & V = -\cos u \end{matrix} \right| = -\left[-u\cos u + \int \cos u\,du\right] = u\cos u - \sin u =$$

$$= \ln x\cdot\cos(\ln x) - \sin(\ln x)$$

similarly,

$$A_2(x) = \int A_2'(x)\,dx = u\sin u + \cos u = \ln x\cdot\sin(\ln x) + \cos(\ln x)$$

Hence,

$$y_p(x) = A_1(x)\cos(\ln x) + A_2(x)\sin(\ln x) =$$

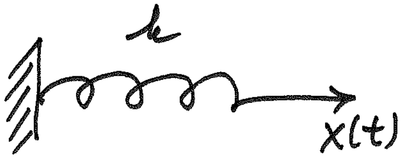
$$= \left[\ln x \cdot \cos(\ln x) \cancel{- \sin(\ln x)}\right]\cos(\ln x) + \left[\ln x \cdot \sin(\ln x) \cancel{+ \cos(\ln x)}\right]\sin(\ln x) =$$

$$= \ln x\,(\underbrace{\cos^2(\ln x) + \sin^2(\ln x)}_{=1}) = \ln x \quad \Rightarrow \quad \boxed{y_p(x) = \ln x}$$

General solution is

$$y(x) = \underbrace{C_1\cos(\ln x) + C_2\sin(\ln x)}_{y_c} + \underbrace{\ln x}_{y_p}$$

#19 S3.6 A mass weighing 100 lb (mass $m = 3.125$ slugs in fps units) is attached to the end of a spring that is stretched 1 in. by a force of 100 lb. A force $F_0 \cos \omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillations occur? Neglect damping.



$F(x) = kx$: Hooke's law

$F = 100$ lb

$1 \text{ in} = \frac{1}{12}$ ft

$\Rightarrow k = \frac{F}{x} = \frac{100}{\frac{1}{12}} = 1200$ lb/ft

$m\ddot{x} + kx = F_0 \cos \omega t$

$W = mg$

$m = \frac{W}{g} = \frac{100}{32} = 3.125$ (slug)

$\omega_0 = \sqrt{\frac{k}{m}}$: natural frequency (in radians)

$\omega_0 = \sqrt{\frac{1200}{3.125}} = \sqrt{384} = 19.6$ (rad/sec) Resonance occurs when $\omega = \omega_0$.

$T = \frac{2\pi}{\omega_0}$: period

$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$: frequency in hertz

$\nu = \frac{\omega_0}{2\pi} = \frac{\sqrt{384}}{2\pi} = 3.12$ (hertz/sec)

#24
S3.6

A mass on a spring without damping is acted on by the external force $F(t) = F_0 \cos^3 \omega t$. Show that there are two values of ω for which resonance occurs, and find both.

$$m\ddot{x} + kx = F_0 \cos^3 \omega t$$

$$\cos^3 \omega t = \frac{1}{4}(3\cos \omega t + \cos 3\omega t)$$

$$\pm i\omega \qquad \pm i3\omega$$

$$m\ddot{x} + kx = \frac{F_0}{4}(3\cos \omega t + \cos 3\omega t)$$

$\omega_0 = \sqrt{\frac{k}{m}}$: natural frequency

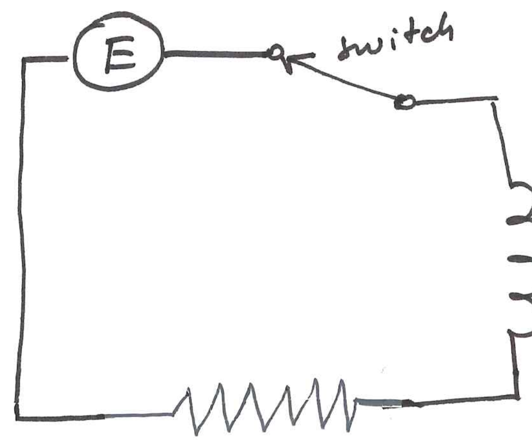
Resonance occurs when either $\boxed{\omega = \omega_0}$ or $3\omega = \omega_0$, i.e. $\boxed{\omega = \frac{\omega_0}{3}}$.

$$\cos^2 \omega t = \frac{1 + \cos 2\omega t}{2}$$

$$\sin^2 \omega t = \frac{1 - \cos 2\omega t}{2}$$

#4 S3.7

In the circuit



$L = 2\ H$

$R = 40\ \Omega$

$E(t) = 100e^{-10t}$, $I(0) = 0$

Find the maximum current in the circuit for $t \geq 0$.

$LI' + RI + \cancel{\frac{1}{C}Q} = E(t)$ (no capacitor)

$2I' + 40I = 100e^{-10t}$

$I' + 20I = 50e^{-10t}$ (1)

$(D+20)I = 50e^{-10t}$

-20 $\qquad -10$

Higher order DE is

$(D+20)(D+10)I = 0$

$-20; -10$

$I(t) = \underbrace{Ce^{-20t}}_{I_c} + \underbrace{Ke^{-10t}}_{I_g}$

Only Now we can use IC to find C

$A(D) = D+10$

To find K, substitute $I_g = Ke^{-10t}$ into (1).

$$\begin{array}{lll} (20) & I_g & Ke^{-10t} \\ (1) \cdot & I_g' & -10Ke^{-10t} \end{array} +$$

$(20-10)Ke^{-10t} = 50e^{-10t} \Rightarrow 10K = 50 \Rightarrow K = 5$

$\Rightarrow \boxed{I_p(t) = 5e^{-10t}}$

$I(t) = Ce^{-20t} + 5e^{-10t}$

$I(0) = 0 \Rightarrow I(0) = C + 5 = 0 \Rightarrow C = -5$

$I(t) = -5e^{-20t} + 5e^{-10t}$

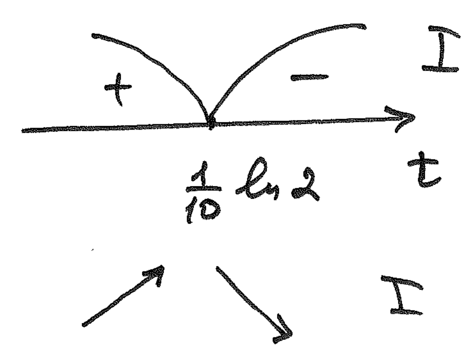
$$I'(t) = -5(-20)e^{-20t} - 5 \cdot 10 e^{-10t} = 100 e^{-20t} - 50 e^{-10t}$$

$$I'(t) = 0 \Rightarrow \underbrace{50 e^{-10t}}_{\neq 0}\left(2e^{-10t} - 1\right) = 0$$

$$\Rightarrow 2e^{-10t} = 1$$

$$e^{-10t} = \frac{1}{2} \quad \text{or} \quad e^{10t} = 2 \Rightarrow 10t = \ln 2$$

$$t = \frac{1}{10}\ln 2$$



$$I_{max} = I\left(\frac{1}{10}\ln 2\right)$$

#17
§3.7

$R = 16\,\Omega$, $L = 2H$, $C = 0.02\,F$ $\quad E(t) = 100\,V$; $\quad I(0) = 0$, $Q(0) = 5$

Find the current $I(t)$.

$\boxed{LI' + RI + \frac{1}{C}Q = E(t)}$

$\boxed{LQ'' + RQ' + \frac{1}{C}Q = E(t)}$ since $I = Q'$ or $\boxed{LI'' + RI' + \frac{1}{C}I = E'(t)}$

① Use $LQ'' + RQ' + \frac{1}{C}Q = E(t)$, $\quad Q(0) = 5$, $\quad Q'(0) = I(0) = 0$

$\frac{1}{0.02} = \frac{100}{2} = 50$

$2Q'' + 16Q' + 50Q = 100 \quad \Big| \frac{1}{2}$

(2) $\quad Q'' + 8Q' + 25Q = 50$ ← nonhomog. 2nd order DE

$(D^2 + 8D + 25)\,Q = 50$ ← root 0

$$\frac{-8 \pm \sqrt{8^2 - 4\cdot 25}}{2} = \frac{-8 \pm \sqrt{64-100}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$$

$$Q(t) = \underbrace{C_1 e^{-4t}\cos 3t + C_2 e^{-4t}\sin 3t}_{Q_c} + \underbrace{K}_{Q_g}$$

To find K, substitute $Q_g = K$ into (2):

$0 + 8\cdot 0 + 25\cdot K = 50$

$\Rightarrow K = 2$

$\Rightarrow \boxed{Q_p = 2}$

$$Q(t) = C_1 e^{-4t}\cos 3t + C_2 e^{-4t}\sin 3t + 2$$

Now use ICs to find C_1 and C_2.

$Q(0)=5 \Rightarrow \quad C_1+2=5 \Rightarrow \boxed{C_1=3}$

$Q'(t)= C_1(-4)e^{-4t}\cos 3t - 3C_1 e^{-4t}\sin 3t - 4C_2 e^{-4t}\sin 3t + 3C_2 e^{-4t}\cos 3t$

$Q'(0)=I(0)=0 \Rightarrow \quad -4\underbrace{C_1}_{=3} + 3C_2 = 0 \Rightarrow 3C_2 = 12 \Rightarrow \boxed{C_2=4}$

$\therefore \boxed{Q(t) = 3e^{-4t}\cos 3t + 4e^{-4t}\sin 3t + 2}$

then

$I(t) = Q'(t) = e^{-4t}\cos 3t \, \cancel{(-4\underbrace{C_1}_{=3} + 3\underbrace{C_2}_{=4})}^{0} + (-3\underbrace{C_1}_{=3} - 4\underbrace{C_2}_{=4})e^{-4t}\sin 3t =$

$= \boxed{-25e^{-4t}\sin 3t = I(t)}$

(2) Use $LI'' + RI' + \frac{1}{c}I = E'(t)$ $\qquad E(t)=100 \Rightarrow E'=0$

$2I'' + 16I' + 50I = 0$

$I'' + 8I' + 25I = 0$ $\qquad$ roots : $-4 \pm 3i$

$I(t) = C_1 e^{-4t}\cos 3t + C_2 e^{-4t}\sin 3t$

$I(0) = 0$

$I'(0) - ?$ $\qquad$ use $\quad LI' + RI + \frac{1}{c}Q = E(t) \Rightarrow I'(0) = \dfrac{\overbrace{E(0)}^{100} - R\cdot \overbrace{I(0)}^{0} - \frac{1}{c}\overbrace{Q(0)}^{5}}{L}$

$$I'(0) = \frac{100 - 16\cdot 0 - 50\cdot 5}{2} = \frac{100-250}{2} = \frac{-150}{2} = -75$$

$$\Rightarrow \boxed{I'(0) = -75}$$

$$I(0) = 0 \Rightarrow \quad C_1 = 0 \quad \Rightarrow I(t) = C_2 e^{-4t} \sin 3t$$

$$I'(t) = C_2(-4) e^{-4t} \sin 3t + 3C_2 e^{-4t} \cos 3t$$

$$I'(0) = -75 \Rightarrow \quad 3C_2 = -75 \Rightarrow \boxed{C_2 = -25}$$

$$\Rightarrow \boxed{I(t) = -25 e^{-4t} \sin 3t}$$

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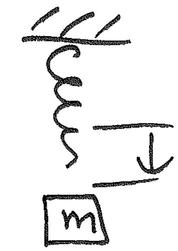
#22
S3.4

A 12-lb weight (mass $m = 0.375$ slugs) is attached both to a vertically suspended spring that it stretches 6 in. and to a dashpot that provides 3 lb of resistance for every foot per second of velocity.

(a) If the weight is pulled down 1 ft below its static equilibrium and then released from rest at $t=0$, find position $x(t)$.

(b) Find frequency, $A(t)$ and phase angle.



$6in = \frac{1}{2}$ ft

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad m = 0.375 \qquad W = 12 \text{ lb}$$

Hooke's law: $F = kx$

$$\underbrace{mg}_{W} = k \cdot x \Rightarrow k = \frac{W}{\frac{1}{2}} = \frac{12}{\frac{1}{2}} = 24$$

$x = 6in = \frac{1}{2}$ ft

$$c = 3$$

$$0.375\ddot{x} + 3\dot{x} + 24x = 0 \qquad 0.375 = \frac{12}{32}$$

$$\left(0.375\,D^2 + 3D + 24\right) x = 0$$

$$\frac{-3 \pm \sqrt{3^2 - 4(0.375)\cdot 24}}{2\cdot(0.375)} = \frac{-3 \pm i\sqrt{27}}{0.75} =$$

$$9 - 4\cdot\frac{12}{32}\cdot 24 = 9 - 12\cdot 3 = 9 - 36 = -27$$

$$\times\begin{array}{r}0.375\\2\\\hline 0.750\end{array}$$

$$= -4 \pm i\cdot 6.93 \quad \text{or} \quad -4 \pm i\cdot 4\sqrt{3}$$

$$0.75 = \frac{3}{4}$$

$$\Rightarrow \frac{\sqrt{27}}{0.75} = \frac{4}{3}\sqrt{27}$$

$$27 = 9\cdot 3$$

$$\Rightarrow \sqrt{27} = 3\sqrt{3}$$

$$\frac{4}{3}\sqrt{27} = \frac{4}{3}\cdot 3\sqrt{3} = 4\sqrt{3}$$

$$x(t) = C_1 e^{-4t}\cos(4\sqrt{3}\,t) + C_2 e^{-4t}\sin(4\sqrt{3}\,t)$$

ICs: $x(0) = 1$ ft (if positive direction is downwards)

$\dot{x}(0) = 0$ (released from rest)

$+\ x$

$$\dot{x}(t) = C_1\cdot(-4)e^{-4t}\cos(4\sqrt{3}\,t) - C_1 e^{-4t}\cdot 4\sqrt{3}\sin(4\sqrt{3}\,t) - 4C_2 e^{-4t}\sin(4\sqrt{3}\,t) + 4\sqrt{3}\,C_2 e^{-4t}\cos(4\sqrt{3}\,t)$$

$$\dot{x}(0) = 0 \Rightarrow \quad -4C_1 + 4\sqrt{3}\,C_2 = 0 \quad \Rightarrow \quad C_1 = \sqrt{3}\,C_2$$

$\frac{4}{3}\sqrt{27} = 4\sqrt{3}$ to simplify solution...

$x(0) = 1 \Rightarrow \quad C_1 = 1$

$\Rightarrow C_2 = \frac{1}{\sqrt{3}} C_1 = \frac{1}{\sqrt{3}}$

$\therefore \; x(t) = e^{-4t} \cos(4\sqrt{3}\, t) + \frac{1}{\sqrt{3}} e^{-4t} \sin(4\sqrt{3}\, t) =$

$= e^{-4t} \left[\cos(4\sqrt{3} t) + \frac{1}{\sqrt{3}} \sin(4\sqrt{3} t) \right] = e^{-4t} A \cdot \cos(4\sqrt{3} t - \delta)$

where $A = \sqrt{C_1^2 + C_2^2} = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}}$

$\tan \delta = \frac{C_2}{C_1} = \frac{1/\sqrt{3}}{1} = \frac{1}{\sqrt{3}}$

$C_1, C_2 > 0 \Rightarrow \delta$ is in I quadrant $\Rightarrow \delta = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\left\{ \begin{array}{l} \sin \frac{\pi}{6} = \frac{1}{2} \\ \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \tan \frac{\pi}{6} = \frac{1/2}{\sqrt{3}/2} \\ \quad = \frac{1}{\sqrt{3}} \end{array} \right.$

$\Rightarrow \quad x(t) = \underbrace{\sqrt{\frac{4}{3}}\, e^{-4t}}_{A(t)} \cos(4\sqrt{3}\, t - \underbrace{\frac{\pi}{6}}_{\text{phase}})$