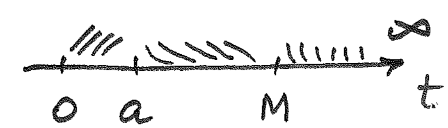
Lecture 34

Recall

Def Laplace transform of a function $f(t)$ is

$$\mathcal{L}\{f(t)\} \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t)\,dt$$

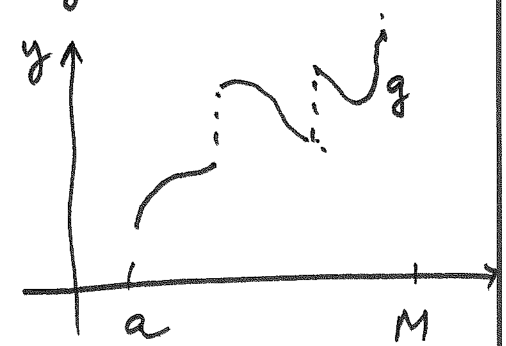
Not all functions have Laplace transforms.



For a function $f(t)$, Laplace transform exists if $\int_0^\infty e^{-st} f(t)\,dt$ <u>converges</u> $(< \infty)$ $(=\text{finite } \#)$

$$\int_0^\infty \ldots = \int_0^a \ldots + \int_a^M \ldots + \int_M^\infty \ldots$$

If each integral on the right exists, then the integral on the left will also exist.



g is piecewise continuous

$\int_a^M g(t)\,dt$: both limits a, M are finite, if $g(t)$ is piecewise continuous, then $\int_a^M g(t)\,dt$ exists

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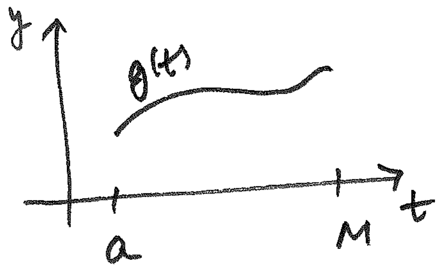
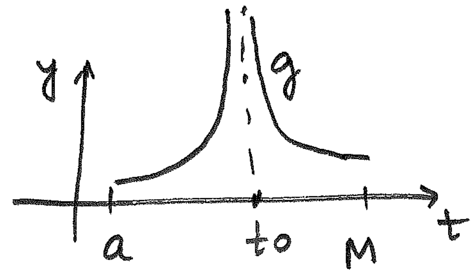


g is continuous

$\Rightarrow$ piecewise continuous



$g(t)$ is not piecewise continuous since it approaches ∞ as $t \to t_0 \in [a, M]$

$\int_0^a h(t)\,dt$: it is possible that even if $h(t) \to \infty$ as $t \to 0$, $\int_0^a h(t)\,dt$ converges

$\int_0^1 \frac{dt}{t^p} < \infty$ if $0 < p < 1$ p-test

Ex $\int_0^1 \frac{dt}{\sqrt{t}}$: exists

$\int_0^1 \frac{dt}{t}$: diverges

If $\int_0^a h(t)\,dt < \infty$, (converges) we say that $h(t)$ is <u>integrable at $t=0$.</u>

$\int_M^\infty y(t)\,dt$. One can show that if $|y(t)| \le Ke^{\alpha t}$ when $t \geq M$ for some $K, \alpha > 0$, then $\int_M^\infty e^{-st} f(t)\,dt$ exists for $s > \alpha$.

For example, $|t^2| \le 2e^t$ for $t \geq 6$

Here $K=2$, $\alpha=1$, $M=6$

$$\Rightarrow \int_6^\infty t^2 e^{-st}\,dt < \infty \quad \text{for } s > \underset{=\alpha}{1}$$

Ex $\left|e^{4t}\sin(t^2+3t)\right| \le 1 \cdot e^{4t}$ $\forall t \geq 0$ $\Rightarrow$ here $K=1$, $\alpha=4$

$$\Rightarrow \int_0^\infty e^{-st} e^{4t} \sin(t^2+3t)\,dt < \infty \quad \text{for } s > 4$$

Def If $|f(t)| \le Ke^{\alpha t}$ for $t \geq M$, some $K>0$, $\alpha>0$, we say that $f(t)$ is of EXPONENTIAL ORDER as $t \to \infty$.

Thm (Existence of Laplace transform)

Given a function $f(t)$ defined on $t>0$, $\mathcal{L}\{f(t)\}$ exists if

(a) $\int_0^a f(t)\,dt$ exists (f(t) is integrable at the origin)

(b) f(t) is picewise continuous on $[a, M]$

(c) f(t) is of exponential order as $t \to \infty$

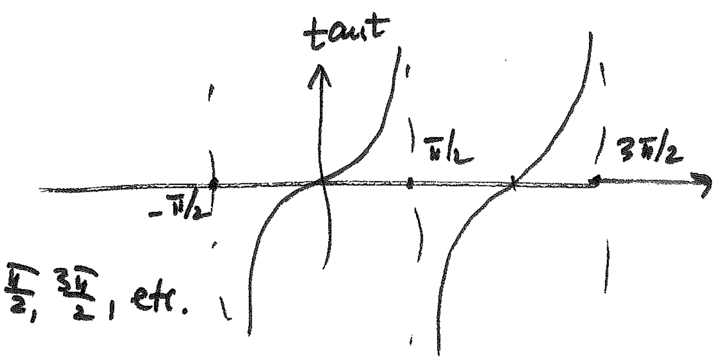
If one of the conditions is not satisfied, the Laplace transform may not exist.

Two examples of functions for which Laplace transform does not exist:

$f(t) = e^{t^2}$ – grows faster than any $e^{\alpha t}$

$\int_0^\infty e^{t^2 - st}\,dt$ does not converge.

$f(t) = \tan t$ – has infinite jumps at $t = \frac{\pi}{2}, \frac{3\pi}{2}$, etc.



<u>Corollary</u> If f(t) satisfies hypotheses of the above thm, then

$$\lim_{s \to \infty} F(s) = 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

<u>Thm</u> If $\mathcal{L}\{f(t)\} = F(s)$ and $f(t)$ is of exponential order as $t \to \infty$, then

$$\boxed{\mathcal{L}\{t\, f(t)\} = -\frac{dF}{ds}} \quad \text{and} \quad \boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}}$$

□ $\mathcal{L}\{f(t)\} \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt \underset{\text{exp order}}{\overset{f \text{ is of}}{=}} \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt =$$

$\Rightarrow$ we can switch the order of $\frac{d}{ds}$ and $\int$

$$\frac{\partial}{\partial s} e^{-st} = -t e^{-st}$$

$$= \int_0^\infty -t e^{-st} f(t)\, dt = -\int_0^\infty e^{-st}\left[t\, f(t)\right] dt = -\mathcal{L}[t\, f(t)]$$

To prove $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$ we can use the method of mathematical induction.

Ex $\mathcal{L}\{1\} = \frac{1}{s}$ $\boxed{\mathcal{L}\{t f(t)\} = -\frac{dF}{ds}}$ $\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}$

$$\mathcal{L}\{t\} = \mathcal{L}\{t \cdot 1\} = -\frac{d}{ds}\mathcal{L}\{1\} = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2} \Rightarrow \boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$$

$$\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = -\frac{d}{ds}\mathcal{L}\{t\} = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3} \Rightarrow \boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}}$$

$$\mathcal{L}\{t^3\} = \mathcal{L}\{t \cdot t^2\} = -\frac{d}{ds}\mathcal{L}\{t^2\} = -\frac{d}{ds}\frac{2}{s^3} = \frac{2 \cdot 3}{s^4} \Rightarrow \boxed{\mathcal{L}\{t^3\} = \frac{6}{s^4}}$$

$6 = 1 \cdot 2 \cdot 3 = 3!$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad n \geq 0}$$

Ex $\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds}\mathcal{L}\{\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\frac{2 \cdot 2s(-1)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$

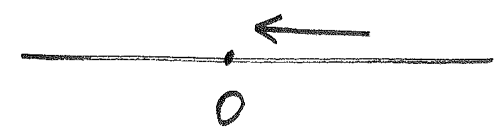
Ex $\mathcal{L}\{t \cos 2t\} = -\frac{d}{ds}\mathcal{L}\{\cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{1 \cdot (s^2+4) - 2s \cdot s}{(s^2+4)^2} =$

$$= \frac{s^2-4}{(s^2+4)^2}$$

One of the reasons we study Laplace transforms is the following property.

<u>Thm</u> If the Laplace transform of a function $f(t)$ is $\mathcal{L}\{f(t)\} = F(s)$, then

$$\boxed{\mathcal{L}\{f'(t)\} = s\,F(s) - f(0^+)}$$



□ $\mathcal{L}\{f'(t)\} \overset{def}{=} \int_0^\infty e^{-st} f'(t)\,dt = \lim_{M\to\infty} \int_0^M e^{-st} f'(t)\,dt \underset{\text{by parts}}{\overset{\text{integr.}}{=}}$

$$= \left| \begin{array}{ll} u = e^{-st} & dv = f'(t)\,dt \\ du = -s e^{-st}\,dt & v = f(t) \end{array} \right| = \lim_{M\to\infty} \left[e^{-st} f(t) \Big|_0^M + s\int_0^M e^{-st} f(t)\,dt \right] = \overset{0}{\lim_{t\to\infty} e^{-st} f(t)} - e^{-st} f(t)\Big|_{t\to 0^+} + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -e^{s\cdot 0} f(0+) + s \underbrace{\mathcal{L}\{f(t)\}}_{F(s)}$$

$$\Rightarrow \quad \mathcal{L}\{f'(t)\} = s\cdot F(s) - f(0^+)$$