Lecture 35

Thm

$0 \quad t$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - \dots$$
$$\dots - s f^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

where $\quad f = f(t) \qquad F = F(s)$

$$F(s) = \mathcal{L}\{f(t)\}$$

e.g. $\boxed{\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0^+) - f'(0^+)}$ $\quad \left(f'(0^+) = \lim_{t \to 0^+} f'(t) \right)$

$$\boxed{\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0^+) - s f'(0^+) - f''(0^+)}$$

Recall

$$\boxed{\mathcal{L}\{f'(t)\} = s F(s) - f(0^+)}$$

Ex Use Laplace transform method to solve

$$y' - 3y = 0, \qquad y(0) = 4$$

Denote $Y(s) = \mathcal{L}\{y(t)\}$. Apply Laplace transform to both sides of the differential equation.

$$\mathcal{L}\{y'(t)\} - 3\underbrace{\mathcal{L}\{y(t)\}}_{Y(s)} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\left(sY(s) - \underbrace{y(0^+)}_{4}\right) - 3Y(s) = 0$$

Solve for $Y(s)$.

$$sY(s) - 4 - 3Y(s) = 0$$

$$(s-3)Y(s) = 4$$

$$Y(s) = \frac{4}{s-3}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Hence,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{4}{s-3}\right\} = 4e^{3t}$$

$\Rightarrow$ $\boxed{y(t) = 4e^{3t}}$ is the solution of $y' - 3y = 0$, $y(0) = 4$

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Ex Solve $y'' + 4y = 0$, $y(0) = 2$, $y'(0) = 8$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Apply Laplace transform to both sides of DE and use ICs.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 0$$

$$\left(s^2 Y(s) - s\underset{2}{y(0)} - \underset{8}{y'(0)}\right) + 4Y(s) = 0$$

$$s^2 Y(s) - 2s - 8 + 4Y(s) = 0$$

Solve for $Y(s)$.

$$(s^2+4)Y(s) = 2s + 8$$

$$Y(s) = \frac{2s+8}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s+8}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\underset{\substack{\uparrow \\ 2\cos 2t}}{\frac{2s}{s^2+4}} + \underset{\substack{\uparrow \\ 4\sin 2t}}{\frac{\overset{4\cdot 2}{\overset{\parallel}{8}}}{s^2+4}}\right\} =$$

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Recall $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$ $\qquad$ $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

$\Leftrightarrow$ $2\cos 2t + 4\sin 2t$

Ex Solve

$y'' - 3y' + 2y = 4, \qquad y(0) = 2, \quad y'(0) = -2$

Apply Laplace transform to both sides of DE. Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$\left(s^2 Y(s) - s\underset{2}{y(0)} - \underset{-2}{y'(0)}\right) - 3\left(sY(s) - \underset{2}{y(0)}\right) + 2Y(s) = \frac{4}{s}$$

$$s^2Y(s) - 2s + 2 - 3(sY(s) - 2) + 2Y(s) = \frac{4}{s}$$

$$(s^2 - 3s + 2)Y(s) = 2s - 2 - 6 + \frac{4}{s}$$

$$(s^2 - 3s + 2)Y(s) = 2s - 8 + \frac{4}{s}$$

$$Y(s) = \frac{2s - 8 + \frac{4}{s}}{s^2 - 3s + 2} \qquad 2s - 8 + \frac{4}{s} = \frac{2s^2 - 8s + 4}{s}$$

$$Y(s) = \frac{2s^2-8s+4}{s(s^2-3s+2)} = \frac{2s^2-8s+4}{s(s-1)(s-2)}$$

Use partial fraction decomposition.

$$\frac{2s^2-8s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

multiply both sides by s and let $s \to 0$

$$\frac{2s^2-8s+4}{(s-1)(s-2)} = A + \cancel{\frac{Bs}{s-1}}^{\,0} + \cancel{\frac{Cs}{s-2}}^{\,0} \quad \Big| \lim_{s\to 0}$$

$$\frac{4}{(-1)(-2)} = A \Rightarrow \boxed{A=2}$$

multiply both sides by $s-1$ and let $s \to 1$.

$$\frac{2s^2-8s+4}{s(s-2)} = \cancel{\frac{A(s-1)}{s}}^{\,0} + B + \cancel{\frac{C(s-1)}{s-2}}^{\,0} \quad \Big| \lim_{s\to 1}$$

$$\frac{2-8+4}{1\cdot(-1)} = B \Rightarrow \boxed{B=2}$$

Multiply both sides by $s-2$ and let $s \to 2$.

$$\frac{2s^2-8s+4}{s(s-1)} = \cancel{\frac{A(s-2)}{s}}^{\,0} + \cancel{\frac{B(s-2)}{s-1}}^{\,0} + C \quad \Big| \lim_{s\to 2}$$

$$\frac{2\cdot 4 - 8\cdot 2 + 4}{2\cdot 1} = C \Rightarrow \boxed{C = -2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

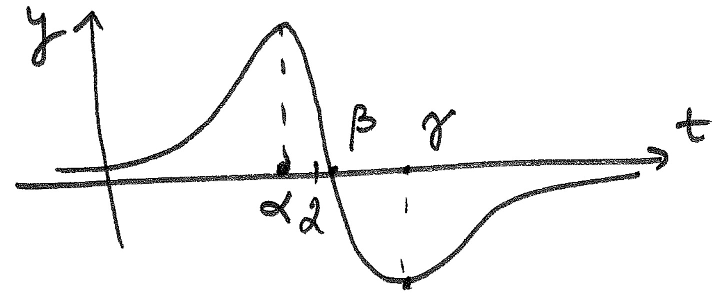
$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2}{s} + \frac{2}{s-1} - \frac{2}{s-2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\Big\{ \underset{\substack{\uparrow \\ 2}}{\frac{2}{s}} + \underset{\substack{\uparrow \\ 2e^{t}}}{\frac{2}{s-1}} - \underset{\substack{\uparrow \\ 2e^{2t}}}{\frac{2}{s-2}} \Big\} = 2 + 2e^{t} - 2e^{2t}$$

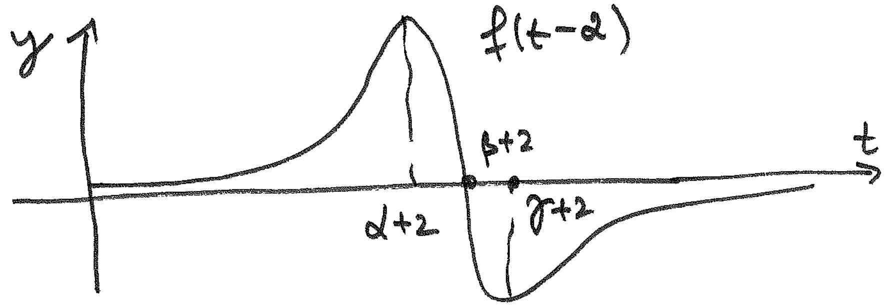
$$\Rightarrow \boxed{y(t) = 2 + 2e^{t} - 2e^{2t}}$$

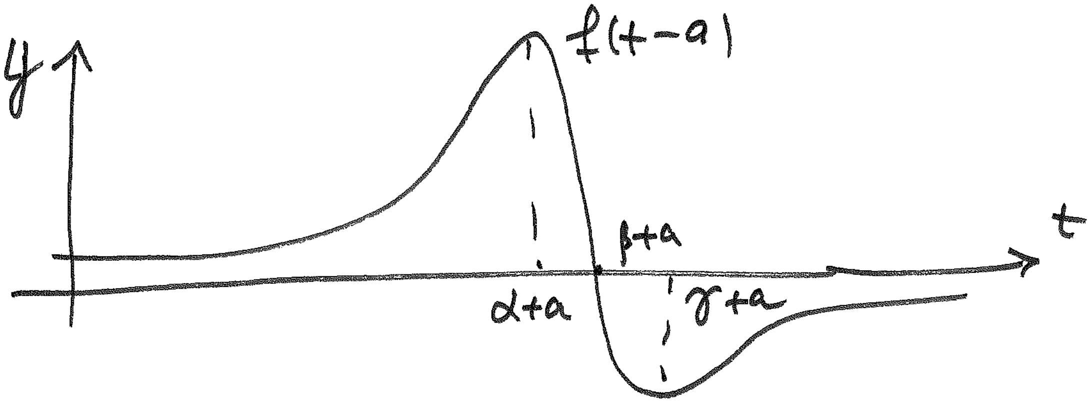
Consider a function $f(t)$ whose graph is given below:



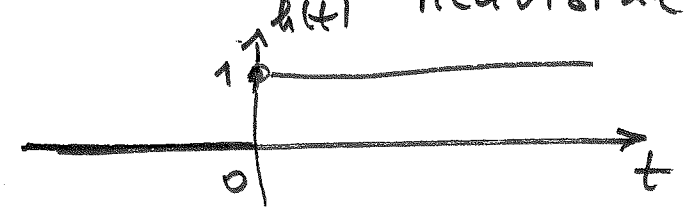
Q How to construct graph of $f(t-2)$ using the graph of $f(t)$?

A We shift the graph of $f(t)$ by 2 units to the right.





STEP FUNCTION (unit step function or Heaviside function)



Def $h(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$

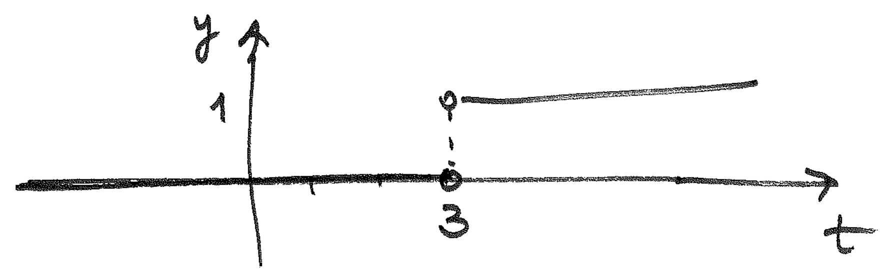
$u(t) = u_0(t)$

$h(t)$ is undefined at $t=0$. We can set $h(0)$ to $0, 1$ or $\frac{1}{2}$.

Observe $h(\alpha) = \begin{cases} 0, & \alpha<0 \\ 1 & \alpha>0 \end{cases}$

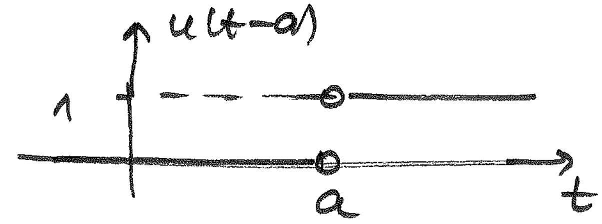
Thus

$h(t-3) = \begin{cases} 0, & \text{if } t-3<0 & \text{or } t<3 \\ 1, & \text{if } t-3>0 & \text{or } t>3 \end{cases}$



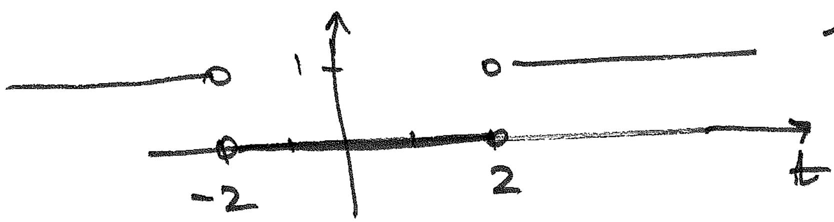
In general,

$$u(t-a) = u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$



Ex Graph $u_0(t^2-4) = h(t^2-4)$

$t^2-4 > 0$ if $t^2 > 4 \Rightarrow t > 2$ or $t < -2$



Ex Graph $h(\sin t)$