Lecture 36

Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 0$$

$$\Gamma(1) = 1, \quad \Gamma(x+1) = x\,\Gamma(x), \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Let n: positive integer

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \dots = \underbrace{n(n-1)(n-2)\dots 2 \cdot 1}_{n!} \cdot \underbrace{\Gamma(1)}_{=1}$$

$$\Rightarrow \quad \boxed{\Gamma(n+1) = n!} \qquad n = 1, 2, 3, \dots$$

Ex

$$\mathcal{L}\{t^a\} \overset{def}{=} \int_0^\infty e^{-st} t^a\, dt = \left| \begin{matrix} u = st \\ t = \frac{u}{s} \end{matrix} \quad dt = \frac{du}{s} \right| =$$

a: real number, $a > -1$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a\, du = \frac{\Gamma(a+1)}{s^{a+1}} \quad \Rightarrow \quad \boxed{\begin{matrix} \mathcal{L}\{t^a\} = \dfrac{\Gamma(a+1)}{s^{a+1}} \\ a\text{: real \#},\ a > -1 \end{matrix}}$$

Let $a = n$: positive integer

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$ derived before

Note

$$\Gamma(x+1) = x\,\Gamma(x)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{\sqrt{\pi}} = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(\tfrac{3}{2}+1\right) = \tfrac{3}{2}\,\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \tfrac{3}{4}\sqrt{\pi} \Rightarrow \boxed{\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}}$$

Ex

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}$$

$$\mathcal{L}\{4t^{3/2}\} = 4\,\mathcal{L}\{t^{3/2}\} =$$

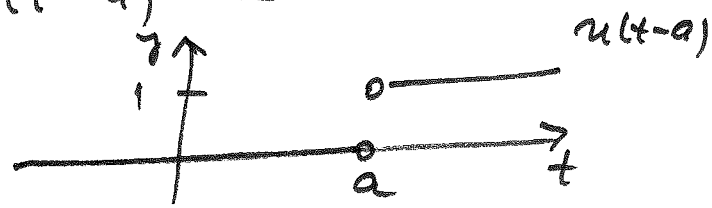
$a = \frac{3}{2}$

$$= 4\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}} = 4\cdot\frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}} = 3\,\frac{\sqrt{\pi}}{s^{5/2}}$$

Step function (Cont'd)

Recall, unit step function $u(t-a)$ is defined

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$



$$\mathcal{L}\{u(t-a)\} \overset{def}{=} \int_0^\infty e^{-st} u(t-a)\,dt = \int_0^a e^{-st} \cancel{u(t-a)}^{0}\,dt + \int_a^\infty e^{-st} \underbrace{u(t-a)}_{=1}\,dt$$

$$= \int_a^\infty e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_{t=a}^{\infty} = -\frac{1}{s}\Big(\cancel{\lim_{t\to\infty} e^{-st}}^{0} - e^{-as}\Big) = \frac{e^{-as}}{s}$$

$s>0$

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$$

Problem

Given $f(t) = \begin{cases} 0, & 0<t<1 \\ 2, & 1<t<3 \\ 0, & t>3 \end{cases}$

find $\mathcal{L}\{f(t)\}$.

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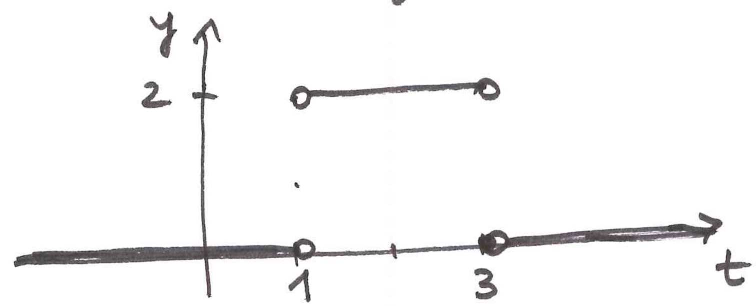
Method I

We can use def of Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} \cancel{f(t)}^{\,0}\,dt + \int_1^3 e^{-st} \underbrace{f(t)}_{=2}\,dt + \int_3^\infty e^{-st} \cancel{f(t)}^{\,0}\,dt$$

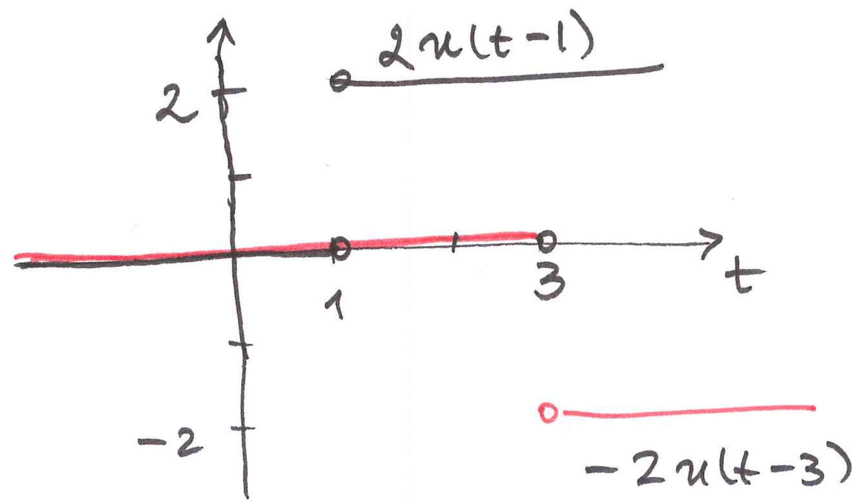
$$= \int_1^3 2e^{-st}\,dt = 2\left.\frac{e^{-st}}{-s}\right|_{t=1}^{t=3} = -\frac{2}{s}\left(e^{-3s} - e^{-s}\right)$$

Method II



$$f(t) = 2u(t-1) - 2u(t-3)$$

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

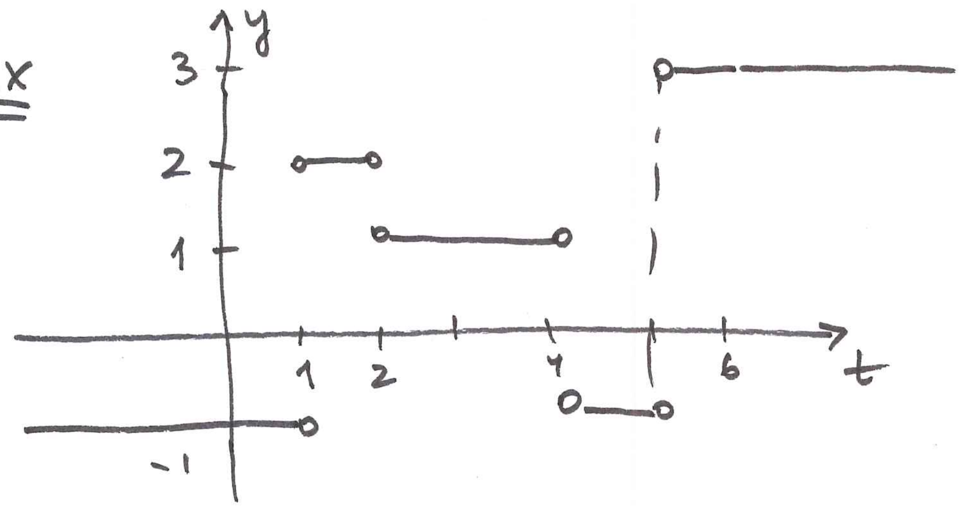


Hence,

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{\underbrace{u(t-1)}_{a=1} - 2\underbrace{u(t-3)}_{a=3}\} = 2\frac{e^{-s}}{s} - 2\frac{e^{-3s}}{s}$$

same as before

Ex



$f(t)$ is given by this graph.
Write $f(t)$ using step functions.

$$f(t) = -1 + 3\,u(t-1) - u(t-2) - 2\,u(t-4) + 4\,u(t-5)$$

$a=1$ $\quad a=2$ $\quad a=4$ $\quad a=5$

$$\mathcal{L}\{f(t)\} = -\frac{1}{s} + 3\frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - 2\frac{e^{-4s}}{s} + 4\frac{e^{-5s}}{s}$$