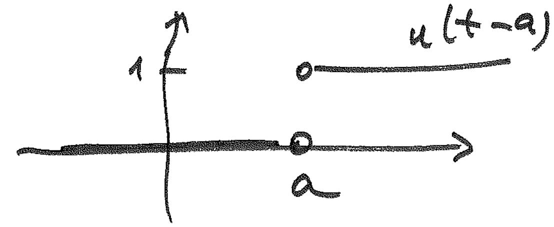
Lecture 37

Recall

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$



<u>Note</u> $\mathcal{L}\{1\} = \frac{1}{s}$

$$\mathcal{L}\{u(t)\} = \underset{a=0}{\mathcal{L}\{u(t-0)\}} = \frac{e^{0}}{s} = \frac{1}{s}$$

Which one is correct? Both.

Recall that Laplace transform is defined for $t \geq 0$:

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\, dt$$

It <u>seems</u> that we can define the inverse Laplace transform for $t<0$ anything we want. One can show that

$$\mathcal{L}^{-1}\{\mathcal{L}\{g(t)\}\} = \begin{cases} g(t), & t \geq 0 \\ 0, & t<0 \end{cases}$$

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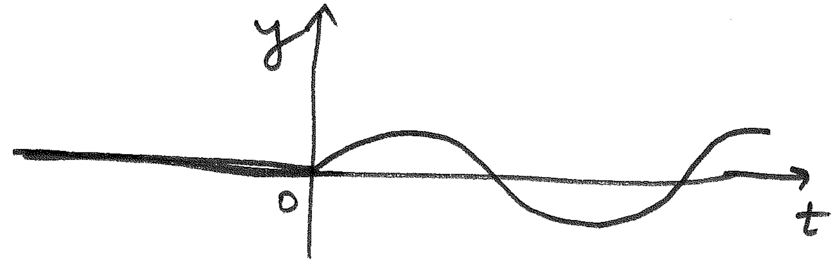
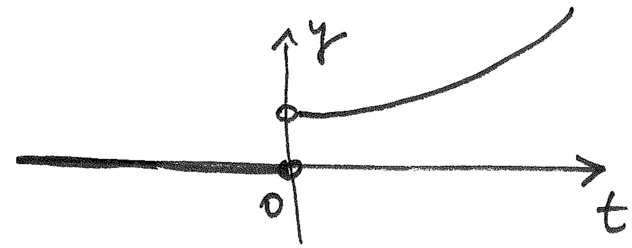
$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \begin{cases} 0, & t<0 \\ \sin t, & t>0 \end{cases}$



$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = \begin{cases} 0, & t<0 \\ e^{t}, & t>0 \end{cases}$



Transforms of Integrals (Section 7.2)

We know that $\int_0^t f(\xi)\,d\xi$ is a function of t.

$\xi\xi \quad \xi\xi$

Q How to find Laplace transform of $\int_0^t f(\xi)\,d\xi$ without evaluating the integral?

Let $g(t) = \int_0^t f(\xi)\,d\xi \qquad g(0)=0$

$g'(t) = f(t) \qquad$ Let $G(s) = \mathcal{L}\{g(t)\}$

$$\mathcal{L}\{g'(t)\} = s\,\underbrace{\mathcal{L}\{g(t)\}}_{G(s)} - \cancel{g(0)}^{\,0} \quad \Rightarrow \quad G(s) = \frac{\mathcal{L}\{g'(t)\}}{s} = \frac{\mathcal{L}\{f(t)\}}{s}$$

$$\therefore \quad \boxed{\mathcal{L}\left\{\int_0^t f(\xi)\,d\xi\right\} = \frac{F(s)}{s}, \quad \text{where} \quad F(s) = \mathcal{L}\{f(t)\}}$$

Ex

$$\mathcal{L}\left\{\int_0^t \underbrace{e^{-\xi}\sin 2\xi}_{f(\xi)}\,d\xi\right\} =$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+2^2}$$

$$\mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+4}$$

$a=-1$

$s \to s-(-1) = s+1$

$$= \frac{\frac{2}{(s+1)^2+4}}{s} = \frac{1}{s}\cdot\frac{2}{(s+1)^2+4}$$

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Ex Solve the following integro-differential equation

$$\frac{dy}{dt} - 3y + 2\int_0^t y(\xi)\,d\xi = 6, \qquad y(0) = 4$$

Apply Laplace transform to both sides of the equation.

Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$\mathcal{L}\left\{\int_0^t y(\xi)\,d\xi\right\} = \frac{Y(s)}{s}$$

$$\Big(sY(s) - \underbrace{y(0)}_{4}\Big) - 3Y(s) + 2\frac{Y(s)}{s} = \frac{6}{s}$$

$$\left(s - 3 + \frac{2}{s}\right)Y(s) = \frac{6}{s} + 4$$

$$Y(s) = \frac{\frac{6}{s} + 4}{s - 3 + \frac{2}{s}} = \frac{4s+6}{s^2 - 3s + 2} = \frac{4s+6}{(s-1)(s-2)} \underset{\text{fraction decomposition}}{\overset{\text{partial}}{=}} \frac{-10}{s-1} + \frac{14}{s-2}$$

$$\therefore \boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = -10e^{t} + 14e^{2t}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Convolution (Section 7.4)

Ex $X(s) = \dfrac{s}{(s^2+1)^2} = \dfrac{s}{s^2+1} \cdot \dfrac{1}{s^2+1} = \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{\sin t\}$

$\mathcal{L}\{\cos t \cdot \sin t\} = \mathcal{L}\{\frac{1}{2}\sin 2t\} = \frac{1}{2}\dfrac{2}{s^2+2^2} = \dfrac{1}{s^2+4}$

$\therefore \quad \mathcal{L}\{\cos t \cdot \sin t\} \neq \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{\sin t\}$

We need an operation that will give us product of Laplace transforms.

Def Convolution of two functions $f(t)$ and $g(t)$ is the function $h(t)$ defined by

$$h(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = f(t) * g(t)$$

Ex $\underset{f}{1} * \underset{g}{1} = \int_0^t 1 \cdot 1\, d\tau = \int_0^t d\tau = \tau \Big|_{\tau=0}^{\tau=t} = t$

Ex

$$\underset{f}{t^3} * \underset{g}{t} = \int_0^t \underset{f(\tau)}{\tau^3} \cdot \underset{g(t-\tau)}{(t-\tau)}\, d\tau = \int_0^t \left(t\cdot\tau^3 - \tau^4\right) d\tau =$$

$$= t\int_0^t \tau^3\, d\tau - \int_0^t \tau^4\, d\tau = t \cdot \frac{\tau^4}{4}\Big|_{\tau=0}^{t} - \frac{\tau^5}{5}\Big|_{\tau=0}^{t} =$$

$$= \frac{t^5}{4} - \frac{t^5}{5} = \frac{t^5}{20}$$

Thm $f * g = g * f$, i.e.

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = g(t) * f(t)$$

Proof

$$\int_0^t f(\tau)\, g(t-\tau)\, d\tau = \left| \begin{array}{l} t-\tau = \omega \\ -d\tau = d\omega \\ \tau = 0 \Rightarrow \omega = t \\ \tau = t \Rightarrow \omega = 0 \end{array} \right| = -\int_t^0 f(t-\omega)\, g(\omega)\, d\omega =$$

$$= \int_0^t g(w)\, f(t-w)\, dw = g(t) * f(t)$$

Thm $(f * g) * h = f * (g * h)$

Thm $f * (g + h) = f * g + f * h$

Convolution Thm If $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$

Then $\mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s)$

and $\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t)$

Thus, the Laplace transform of $f * g$ is the product of $F(s)$ and $G(s)$.

$*$ in time domain $=$ $\cdot$ in s domain

<u>Ex</u> Find $\mathcal{L}^{-1}\{\frac{1}{s^2}F(s)\}$ if $F(s) = \mathcal{L}\{f(t)\}$

$\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$; $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$$\Rightarrow \mathcal{L}^{-1}\{\frac{1}{s^2}\cdot F(s)\} = t * f(t) = \int_0^t \tau \cdot f(t-\tau)\,d\tau$$

If we knew $f(t)$, we would be able to evaluate the integral.

<u>Ex</u> Solve

$$y'' + y = f(t), \quad y(0)=2, \quad y'(0)=4$$

where $f(t)$ is some function of t.

<u>Solution</u>

Apply Laplace transform to both sides of DE.

Let $Y(s) = \mathcal{L}\{y(t)\}$, $F(s) = \mathcal{L}\{f(t)\}$

$$s^2 Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{4} + Y(s) = F(s)$$

$$(s^2+1)Y(s) = F(s) + 2s + 4$$

$$Y(s) = \frac{F(s)}{s^2+1} + \frac{2s+4}{s^2+1} = \frac{1}{s^2+1} \cdot F(s) + \frac{2s}{s^2+1} + \frac{4}{s^2+1}$$

$$y(t) = \underbrace{\sin t * f(t)}_{\int_0^t \sin\tau \cdot f(t-\tau)\,d\tau} + 2\cos t + 4\sin t$$

$$\therefore \quad \boxed{y(t) = \int_0^t \sin\tau \cdot f(t-\tau)\,d\tau + 2\cos t + 4\sin t}$$