Lecture 38

Recall

Thm Differentiation of Transforms

If $f(t)$ is piecewise continuous for $t \geq 0$ and $|f(t)| \leq Me^{ct}$ as $t \to \infty$ (f is of exponential order as $t \to \infty$)

then

$$\boxed{\mathcal{L}\{t\, f(t)\} = -\underbrace{\frac{dF}{ds}}_{=-F'(s)}}, \quad \text{where} \quad F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

equivalently,

$$\boxed{f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}}$$

More generally,

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}}$$

Ex Find inverse Laplace transform of

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S7.4

$$F(s) = \ln \frac{s^2+1}{(s+2)(s-3)}$$

$\ln \frac{a}{b} = \ln a - \ln b$

Solution

$$F(s) = \ln \frac{s^2+1}{(s+2)(s-3)} = \ln(s^2+1) - \ln(s+2) - \ln(s-3)$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{1}{s+2} - \frac{1}{s-3}$$

: it is easier to find inverse Laplace transform of $F'(s)$ than of $F(s)$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\mathcal{L}^{-1}\{F'(s)\} = 2\cos t - e^{-2t} - e^{3t}$$

Recall

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\,\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\left(2\cos t - e^{-2t} - e^{3t}\right)$$

Ex Find a nontrivial solution of

#31 §7.4

$$t x'' - (4t+1) x' + 2(2t+1) x = 0, \qquad x(0) = 0$$

2nd order DE, linear, homog w/ variable coefficients

$$t x'' - 4t x' - x' + 4t x + 2x = 0$$

We will apply Laplace transform to both sides of DE.

Let $X(s) = \mathcal{L}\{x(t)\}$.

$$\mathcal{L}\{x'(t)\} = s X(s) - \cancel{x(0)}^{\,0} = s X(s)$$

$$\mathcal{L}\{x''(t)\} = s^2 X(s) - s\cancel{x(0)}^{\,0} - \underbrace{x'(0)}_{\text{const}} = s^2 X(s) - x'(0)$$

$$\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds} F(s)$$

$$\mathcal{L}\{t \cdot x\} = -\frac{d}{ds} X(s)$$

$$\mathcal{L}\{t \cdot x'\} = -\frac{d}{ds} \mathcal{L}\{x'(t)\} = -\frac{d}{ds}\left(s X(s)\right)$$

$$\mathcal{L}\{t \cdot x''\} = -\frac{d}{ds} \mathcal{L}\{x''(t)\} = -\frac{d}{ds}\left(s^2 X(s) - \underbrace{x'(0)}_{\text{const}}\right) = -\frac{d}{ds}\left(s^2 X(s)\right)$$

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Applying Laplace transform to both sides of DE we get

$$-\frac{d}{ds}\left(s^2 X(s)\right) + 4\frac{d}{ds}\left(sX(s)\right) - sX(s) - 4\frac{d}{ds}X(s) + 2X(s) = 0$$

$$-2sX - s^2X' + 4X + 4sX' - sX - 4X' + 2X = 0$$

$$\left(-s^2 + 4s - 4\right)X' + \left(-2s + 4 - s + 2\right)X = 0$$

$$-\underbrace{\left(s^2 - 4s + 4\right)}_{(s-2)^2}X' + \underbrace{\left(-3s + 6\right)}_{-3(s-2)}X = 0 \quad \Big| \ \frac{1}{s-2}$$

$$-(s-2)\frac{dX}{ds} = 3X$$: 1st order separable DE ~~for~~ for $X(s)$

$$\frac{dX}{X} = -\frac{3}{s-2}\,ds$$

$$\ln|X| = -3\ln|s-2| + \tilde{C} \quad \Big| \ \exp$$

$$X(s) = C\,(s-2)^{-3} \quad \text{or} \quad \boxed{X(s) = \frac{C}{(s-2)^3}}$$

$$-3\ln(s-2) = \ln(s-2)^{-3}$$

$$e^{-3\ln(s-2)} = e^{\ln(s-2)^{-3}} = (s-2)^{-3}$$

$s > 2$

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$$\therefore \quad x(t) = \underbrace{\frac{c}{2}}_{\equiv A} t^2 e^{2t} = A t^2 e^{2t}$$

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$X(s) = \frac{c}{s^3} \Rightarrow \mathcal{L}^{-1}\left\{\frac{c}{s^3}\right\} = \frac{1}{2!} c t^2$$

$s-2 \Rightarrow e^{2t}$ in t-space

$$\Rightarrow \frac{1}{2!} c t^2 e^{2t}$$

Thm 3 Integration of transforms

Suppose $f(t)$ is piecewise continuous for $t \geq 0$ and $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists (and finite). Then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\, d\sigma \quad \text{where} \quad F(s) = \mathcal{L}\{f(t)\}$$

equivalently,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t \cdot \mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\, d\sigma\right\}$$

Proof

$$F(\sigma) \overset{\text{def}}{=} \int_0^\infty e^{-\sigma t} f(t)\,dt$$

$$\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \int_0^\infty e^{-\sigma t} f(t)\,dt\,d\sigma \underset{\substack{\text{the order}\\ \text{of integration}}}{\overset{\text{switch}}{=}} \int_0^\infty \int_s^\infty e^{-\sigma t} f(t)\,d\sigma\,dt$$

(can do because $f(t)$ is of exp order)

$$= \int_0^\infty \frac{e^{-\sigma t}}{-t} f(t)\Big|_{\sigma=s}^{\infty} dt = \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}. \blacksquare$$

Ex
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S7.4

$$f(t) = \frac{1-\cos 2t}{t}$$ Find $\mathcal{L}\{f(t)\}$

Check $\lim_{t\to 0^+} \frac{1-\cos 2t}{t}$ = exists?

$$\lim_{t\to 0^+} \frac{1-\cos 2t}{t} \underset{\text{rule}}{\overset{\text{l'Hopital}}{=}} \lim_{t\to 0^+} \frac{2\sin 2t}{1} = 0 :$$ finite ✓

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

Hence

$$\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \mathcal{L}\{1-\cos 2t\}\,d\sigma =$$

$$\mathcal{L}\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$= \int_s^\infty \left(\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+4}\right) d\sigma =$$

$$\int \frac{\sigma\,d\sigma}{\sigma^2+4} = \left| \begin{array}{l} u = \sigma^2+4 \\ du = 2\sigma\,d\sigma \end{array} \right| = \int \frac{\frac{1}{2}\,du}{u}$$

$$= \left(\ln\sigma - \frac{1}{2}\ln(\sigma^2+4)\right)\Big|_{\sigma=s}^{\infty} =$$

$$= \ln\frac{\sigma}{(\sigma^2+4)^{1/2}}\Big|_{\sigma=s}^{\infty} = \underbrace{\lim_{\sigma\to\infty} \ln\frac{\sigma}{(\sigma^2+4)^{1/2}}}_{\ln \underbrace{\lim_{\sigma\to\infty}\frac{\sigma}{(\sigma^2+4)^{1/2}}}_{=1} = \cancel{\ln 1}^{\,0}}$$

$$- \frac{s}{(s^2+4)^{1/2}} = -\frac{s}{(s^2+4)^{1/2}}$$

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Ex Find $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\}$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

$$f(t) = t \cdot \mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$

$$\int_s^\infty \frac{2\sigma}{(\sigma^2-1)^2}\,d\sigma = \left|\begin{array}{l} u=\sigma^2-1 \\ du = 2\sigma\,d\sigma \\ \sigma = s \Rightarrow u = s^2-1 \end{array}\right|$$

$$= \int_{s^2-1}^\infty \frac{du}{u^2} = -\frac{1}{u}\Big|_{u=s^2-1}^{\infty} = \frac{1}{s^2-1}$$

$$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\cdot\sinh t$$