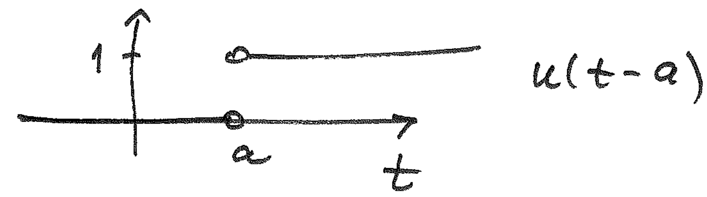
Lecture 39

(Section 7.5)

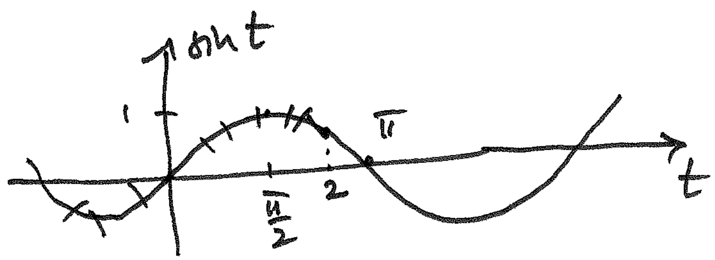
Recall

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

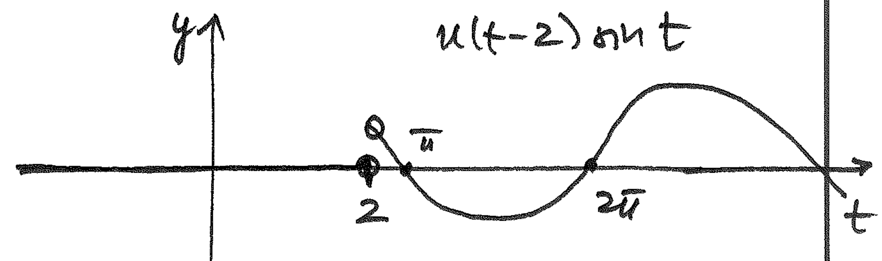


Graph function $u(t-2)\sin t$.

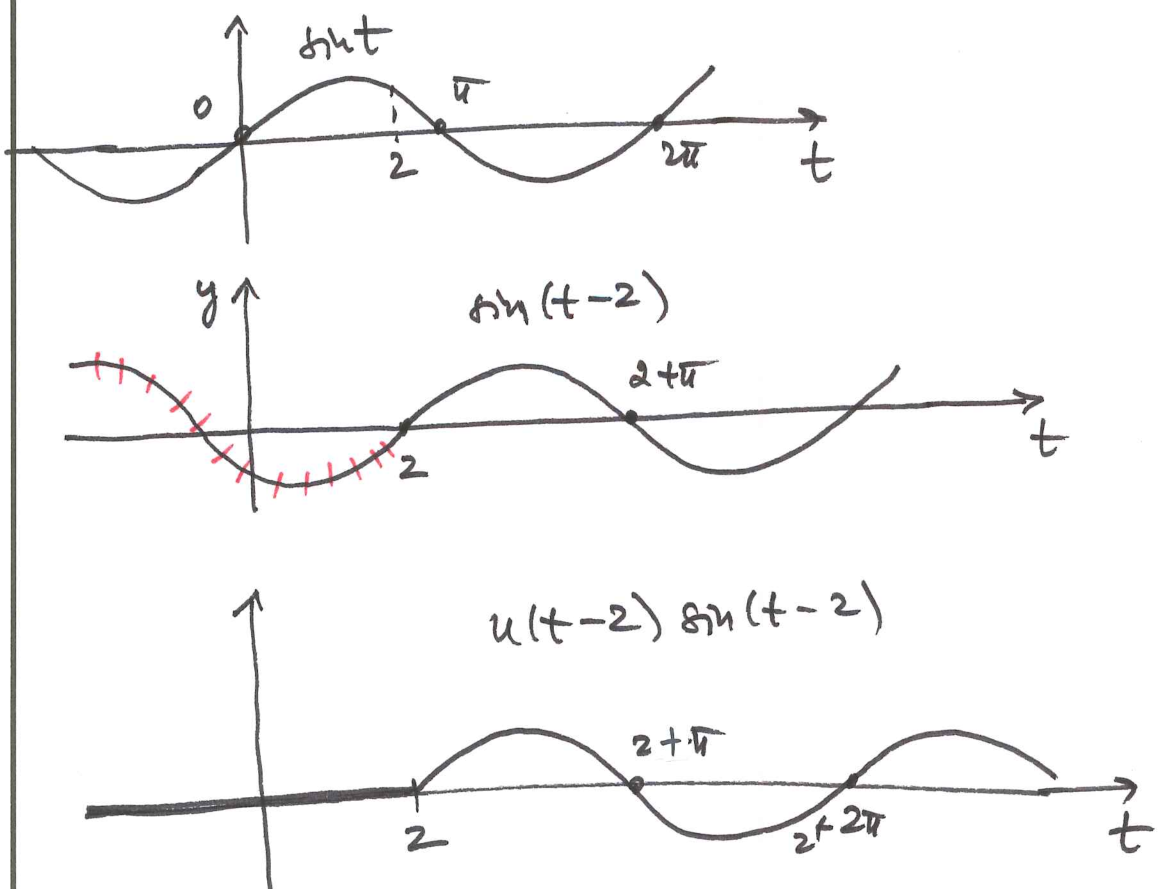
The easiest way is to graph $\sin t$ and then "erase" (make zero or multiply by zero) the graph of $\sin t$ to the left from $t=2$ and leave the graph unaltered (multiply by 1) for $t > 2$.



$\Rightarrow$



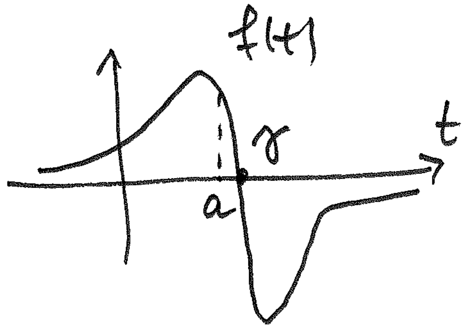
Graph $u(t-2)\sin(t-2)$. $\sin(t-2)$ resembles $\sin t$ but it is shifted by 2 units to the right. Then $u(t-2)$ "erases" the graph for $t<2$.

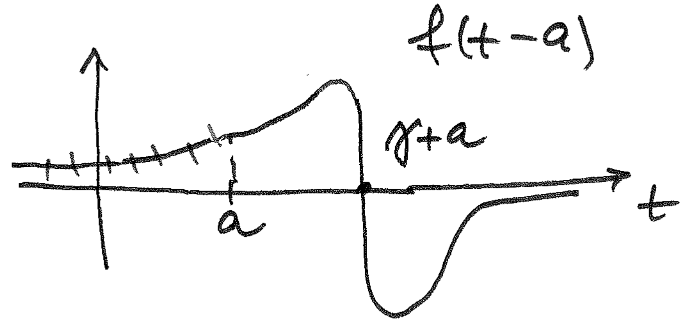


Because of $u(t-2)$, we "chop off" the graph to the left of 2

Q What is the graph of $u(t-a)\,f(t-a)$?

A $u(t-a)\,f(t-a)$ looks like $f(t)$ but shifted by a units to the right, i.e. it is $f(t-a)$ for $t>a$, and it is zero for $t<a$





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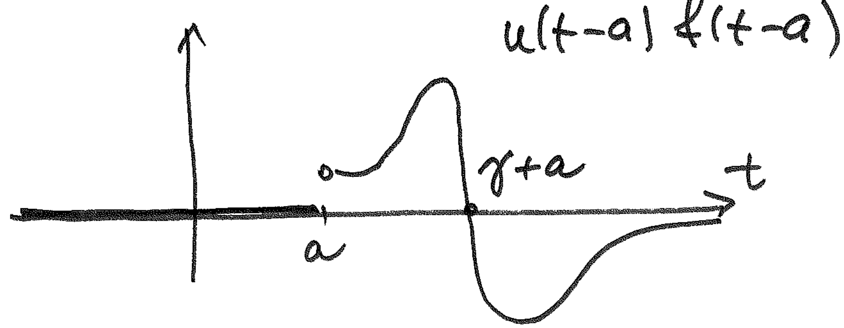

Thm (Shift-Chop Thm)

$$\mathcal{L}\{u(t-a)\, f(t-a)\} = e^{-as} F(s), \quad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

i.e. Laplace transform of shifted and "chopped" function equals Laplace transform of the original function (not shifted) multiplied by e^{-as}.

Problem Graph $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

$e^{-2s} \cdot \frac{1}{s^2}$

$a=2$

$\frac{1}{s^2} = \mathcal{L}\{t\} \Rightarrow f(t) = t$

$t \to t-a = t-2$

$u(t-2)$

$$\mathcal{L}\{u(t-a)\, f(t-a)\} = e^{-as} F(s)$$

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, f(t-a)$$

Recall

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\therefore \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s^2}\right\} = (t-2) \cdot u(t-2)$$

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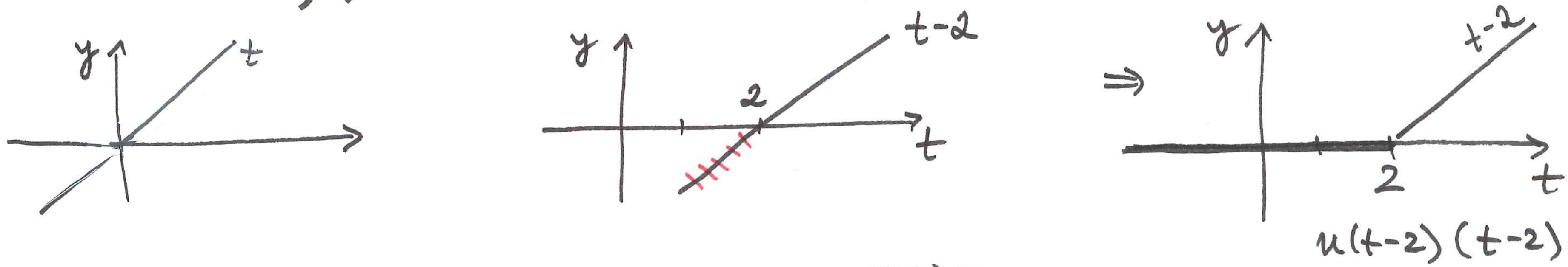
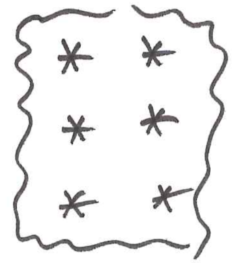


<u>Ex</u> Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}(2s+1)}{s^2+2s+5}\right\}$



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$F(s)=\dfrac{2s+1}{s^2+2s+5} =$

In this case we complete the square:

$$s^2+2s+5 = s^2+2s+1+4 = (s+1)^2+2^2$$

$$= \frac{2s+1}{(s+1)^2+2^2} = \frac{2(s+1)-2+1}{(s+1)^2+2^2} =$$

You can apply partial fraction decomposition to quotient of polynomials <u>only</u>!

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$= \frac{2(s+1)-1}{(s+1)^2+2^2} = \frac{2(s+1)}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}$$

$\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$ $\qquad s+1 \Rightarrow e^{-t}$ $\qquad a=-1$ $\qquad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\therefore$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right\} =$$

$$2\cos 2t \cdot e^{-t} \qquad -\frac{1}{2}\sin 2t \cdot e^{-t}$$

$\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

$$= 2\cos 2t \cdot e^{-t} - \frac{1}{2}\sin 2t \cdot e^{-t} = f(t)$$

Recall Shift-Chop Thm: $\quad \mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, f(t-a)$

$$\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{2s+1}{s^2+2s+5}\right\} = \left[2\cos 2(t-1) \cdot e^{-(t-1)} - \frac{1}{2}\sin 2(t-1) \cdot e^{-(t-1)}\right] u(t-1)$$

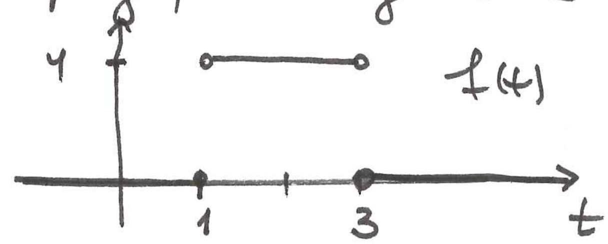
$a=1 \Rightarrow t \to t-1$

$, u(t-1)$

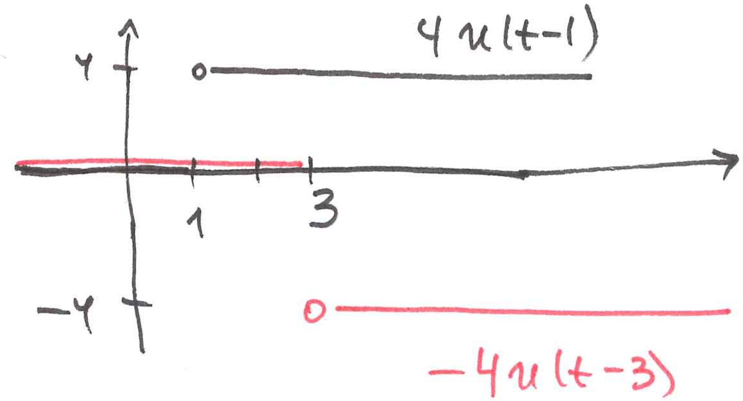
Problem Mass-spring system with damping, driving force $f(t)$, satisfies the ODE

$$\ddot{x} + 3\dot{x} + 2x = f(t)$$

$$x(0) = 2, \quad \dot{x}(0) = -1$$



Note $f(t) = 4u(t-1) - 4u(t-3)$



Apply Laplace transform to both sides of DE.

Recall $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$\Big(s^2 X(s) - s\underbrace{x(0)}_{2} - \underbrace{x'(0)}_{-1}\Big) + 3\Big(s X(s) - \underbrace{x(0)}_{2}\Big) + 2X(s) = 4\frac{e^{-s}}{s} - 4\frac{e^{-3s}}{s}$$

$$s^2X(s)-2s+1 \;+\; 3\left(sX(s)-2\right)+2X(s)=4\frac{e^{-s}}{s}-4\frac{e^{-3s}}{s}$$

$$\left(s^2+3s+2\right)X(s)=\underbrace{2s-1+6}_{2s+5}+4\frac{e^{-s}}{s}-4\frac{e^{-3s}}{s}$$

$$X(s)=\frac{2s+5}{s^2+3s+2}+e^{-s}\cdot\frac{4}{s(s^2+3s+2)}-e^{-3s}\cdot\frac{4}{s(s^2+3s+2)}$$

keep the terms with e^{-s} and e^{-3s} separately, do not mix them!