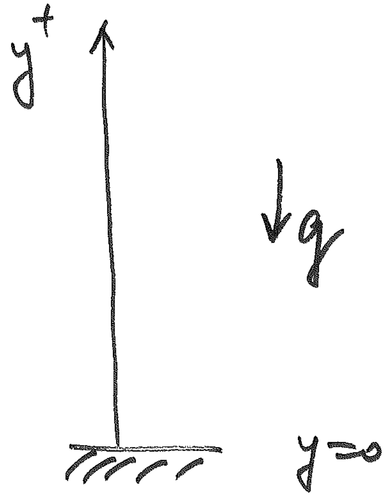
Lecture 3

Vertical Motion with Gravitational Acceleration

Let $\bar{W}$: weight of a body: force exerted on the object by gravity

g: acceleration due to gravity

$mg = \bar{W} \Rightarrow \boxed{m = \frac{\bar{W}}{g}}$



$y(t)$: displacement

$v(t) = \frac{dy}{dt}$

$\frac{dv}{dt} = -g$

$a \rightarrow -g$

We can use results from previous lecture and replace a with $-g$.

ICs:

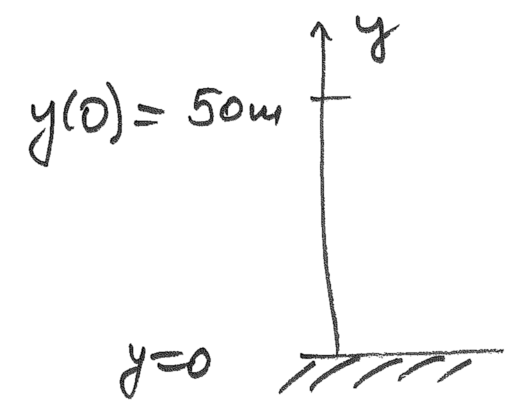
$y(0) = y_0$: initial displacement

$\frac{dy}{dt}(0) = v_0$: initial velocity

$$v(t) = -gt + v_0$$

$$y(t) = -\frac{g}{2}t^2 + v_0 t + y_0$$

<u>Ex</u> A ball is dropped from the top of a building 50 m high. How long does it take for the ball to reach the ground? With which speed does the ball strike the ground?



Here $y(0) = 50$ m : initial position

$v(0) = 0$ m/s : initial velocity

$v(t) = -gt + \cancel{v_0}^{0} = -gt$

$y(t) = -\frac{g}{2}t^2 + \cancel{v_0}^{0} t + \underset{\overset{\shortparallel}{50}}{y_0} = -\frac{g}{2}t^2 + 50$

$g = 9.8 \frac{m}{s^2} \approx 10 \frac{m}{s^2}$

at the ground, $y=0$

$-\frac{g}{2}t^2 + 50 = 0 \Rightarrow t^2 = \frac{2}{g} \cdot 50 = \frac{100}{g}$

$t = \sqrt{\frac{100}{g}} \approx \sqrt{\frac{100}{10}} = \sqrt{10}\ (s)$: time to reach ground

at the ground, $v(t) = v(\sqrt{10}) \approx -g \cdot \sqrt{10} = -10\sqrt{10}\ m/s < 0$

moving downwards

velocity w/ which ball strikes the ground

speed at the ground is

$|v(\sqrt{10})| = 10\sqrt{10}\ m/s$

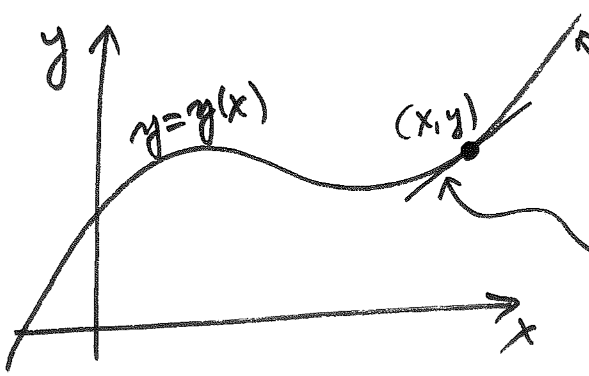
1.3 Slope Fields and Solution Curves

Consider

$$\frac{dy}{dx} = f(x,y)$$

depends on both x and y

In general, it may not be possible to find solution $y(x)$ explicitly. But solution can be approximated <u>numerically</u> or <u>graphically</u>.



solution curve : graph of a solution $y(x)$

slope of the tangent line at (x,y) is given by

$$m = \frac{dy}{dx} = f(x,y) \text{ : given / known function}$$

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at every pt (x,y) we draw a small line segment whose slope $= f(x,y)$. The collection of all such small segments is called a <u>slope field</u> or <u>direction field</u>.

<u>Note</u> If a solution curve goes through pt (x,y), the slope of its tangent line at this point is $f(x,y)$.

<u>Ex</u> $\frac{dy}{dx} = ky$, let $k=1$, i.e. $\frac{dy}{dx} = \underbrace{y}_{f(x,y)}$

$\Rightarrow$ slope at (x,y) is $m = y$

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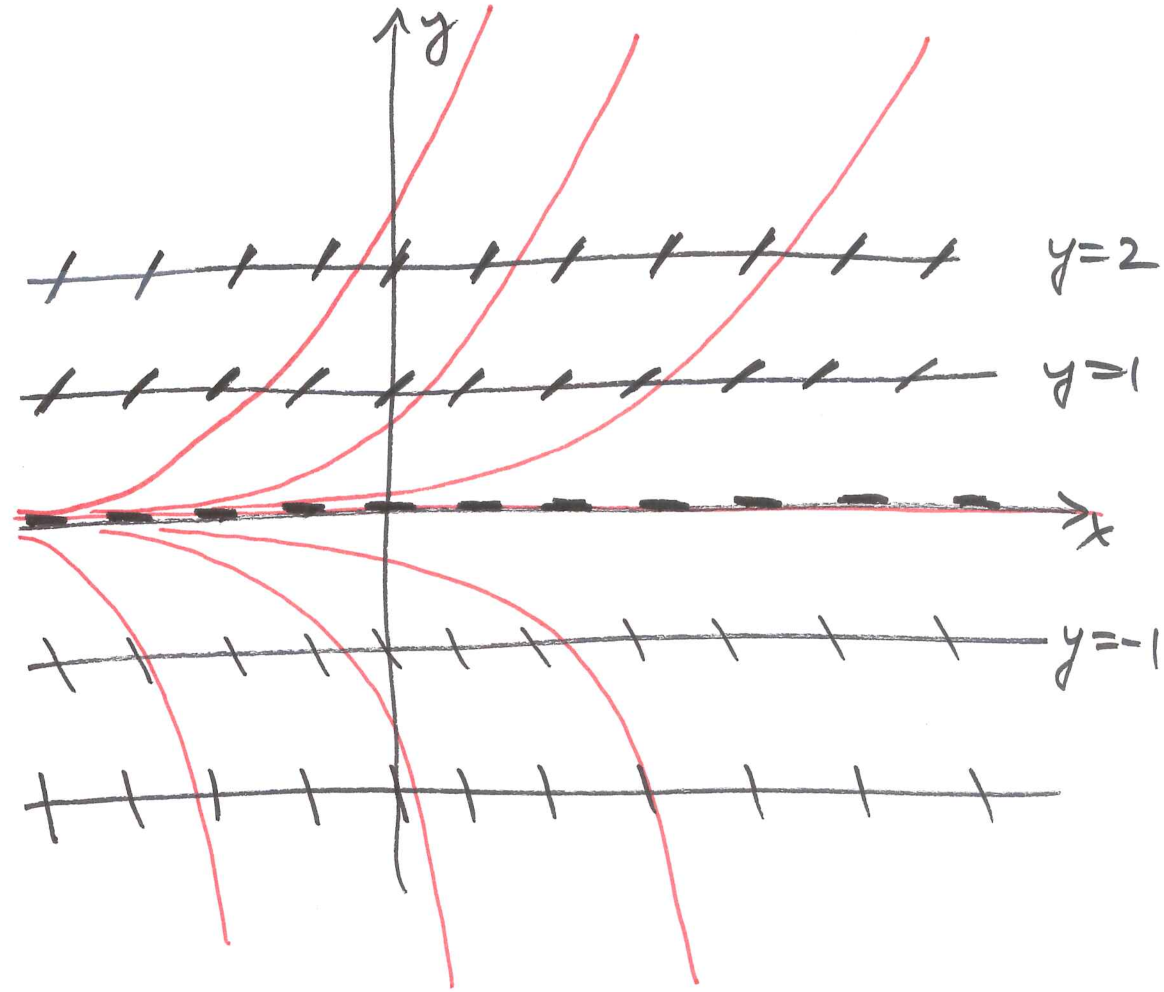
$y=0$ is x-axis

$\Rightarrow m=0$ along x-axis

$y=1 \Rightarrow m=1$: slope of small segments

$y=0$



$y=2 \Rightarrow m=2$

$y=-1 \Rightarrow m=-1$

$y=-2 \Rightarrow m=-2$

$$\frac{dy}{dx} = \underbrace{y}_{m} \quad f(x,y)$$

$$\frac{dy}{dx} = y$$

$y=0$ is a solution

$y'=0 \quad 0=0$

Ex $\frac{dy}{dx} = \underbrace{x-y}_{m = f(x,y)}$

$m=0$ when $x-y=0$

or $y=x$

$y=x+1 \Rightarrow m = x-y =$

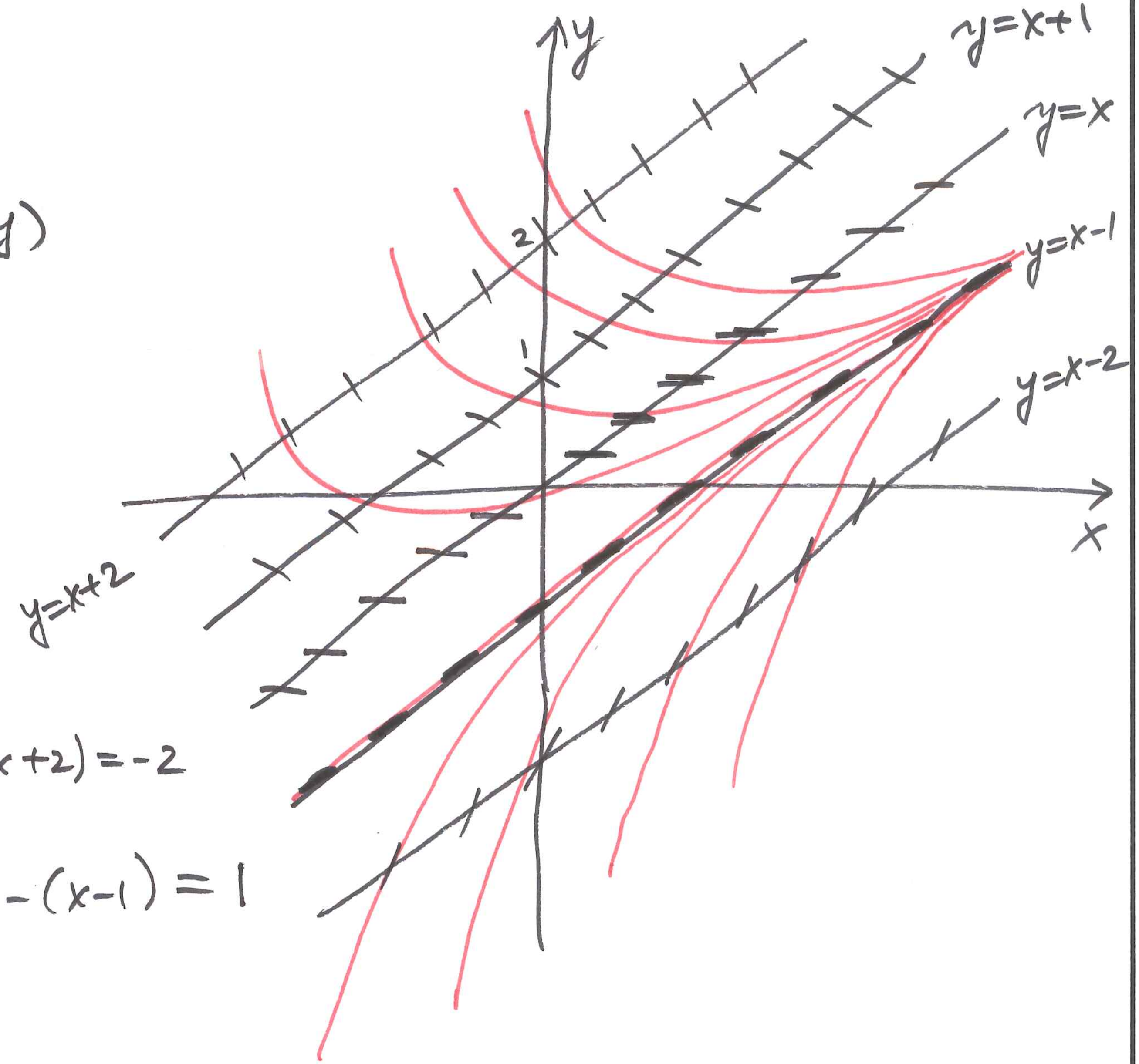
$= x-(x+1) = -1$

$y=x+2 \Rightarrow m = x-y = x-(x+2) = -2$

$y=x-1 \Rightarrow m = x-y = x-(x-1) = 1$

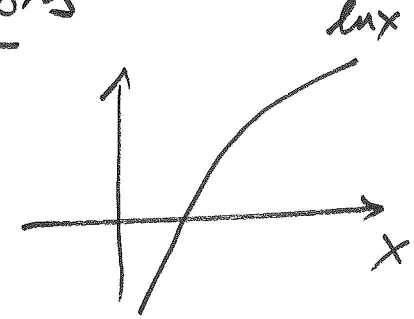
$y=x-2 \Rightarrow m=2$

$y=x-1$ is a solution



$\frac{dy}{dx} = 1$ $\quad x-y = x-(x-1) = 1$

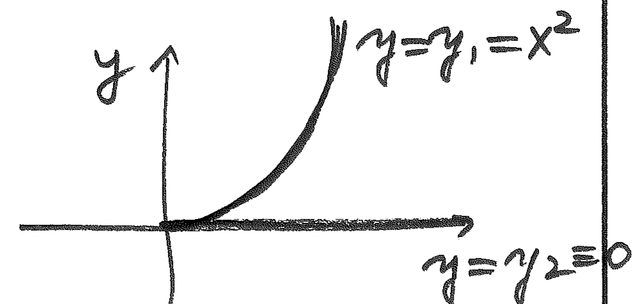
Existence and Uniqueness of Solutions



1) $y' = \frac{1}{x}$, $\quad y(0) = 0$

$y(x) = \int \frac{1}{x}\,dx = \ln|x| + C$: not defined at $x=0$

This IVP does NOT have a solution



2) $y' = 2\sqrt{y}$, $\quad y(0) = 0$, $\quad x > 0$

$y_1(x) = x^2$, $\quad y_2(x) \equiv 0$

This IVP has more than one solution.

Thm (Existence and Uniqueness of Solutions)

Consider

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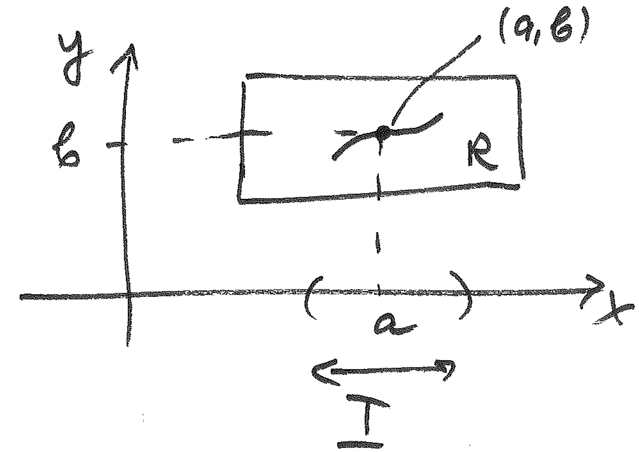

(1) $\dfrac{dy}{dx} = f(x,y) \qquad y(a) = b$

Suppose that both $f(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ are continuous on some rectangular domain R that contains pt (a,b). Then there exist a unique solution $y(x)$ of the IVP (1) on some interval I containing ~~pt~~ a.

1) if $f(x,y)$ is continuous (CTS) on $R \Rightarrow$ solution of IVP exists

2) if $\dfrac{\partial f}{\partial y}$ is CTS on $R \Rightarrow$ solution of IVP is unique

$\dfrac{\partial f}{\partial y}$ is CTS on R implies f is CTS on R