Lecture 3

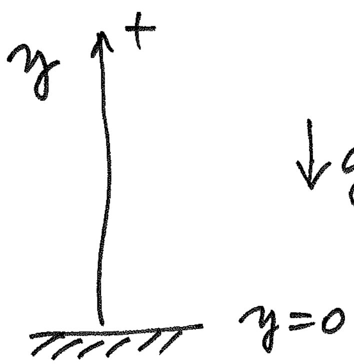
Physical Units

	British	Metric
Force	lb	N
Mass	slug	kg
Distance	ft	m
Time	s	s
g	$32\ ft/s^2$	$9.8\ m/s^2$

Vertical Motion with Gravitational Acceleration

Weight W of a body: force exerted on the object by gravity

$a = g$ $\quad ma = F \Rightarrow mg = W \Rightarrow \boxed{m = \frac{W}{g}}$



$\downarrow g$

$y(t)$: vertical displacement

$$v(t) = \frac{dy}{dt}$$

$$a = \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -g$$

$v(t) = -gt + v_0$

$y(t) = -\frac{g}{2}t^2 + v_0 t + y_0$

ICs: $y(0) = y_0$: initial displacement

$\frac{dy}{dt}(0) = v_0$: initial velocity

Ex A ball is dropped from the top of a building 50 m high. How long does it take for the ball to reach the ground? With which speed does the ball strike the ground?

$y(0) = 50$ m, $v_0 = v(0) = 0$ m/s

$g = 9.8\ m/s^2 \approx 10\ m/s^2$

$v(t) = -gt + \cancel{v_0}^{\,0} = -gt$

$y(t) = -\frac{g}{2}t^2 + \cancel{v_0}^{\,0}t + \underbrace{y_0}_{50} = -\frac{g}{2}t^2 + 50$

At the ground: $y(t) = 0$

$-\frac{g}{2}t^2 + 50 = 0 \Rightarrow t^2 = \frac{2 \cdot 50}{g} = \frac{100}{g}$

$t = \sqrt{\frac{100}{g}} = \frac{10}{\sqrt{g}} \approx \frac{10}{\sqrt{10}} = \sqrt{10}$ (s) : time to reach ground

At the ground, $v(\sqrt{10}) = -g \cdot \sqrt{10} \approx -10\sqrt{10} < 0$ (m/s) : velocity w/ which ball strikes the ground

$\Rightarrow$ speed at ground is $\approx 10\sqrt{10}$ m/s.

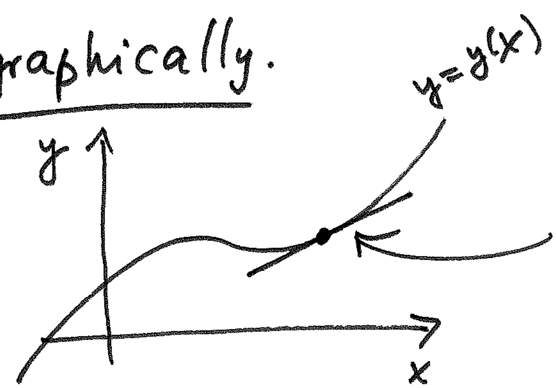
1.3 Slope Fields and Solution Curves

Consider

$$\frac{dy}{dx} = f(x, y)$$

└ depends on y as well

In general, it may not be possible to find $y(t)$ explicitly.
But solution can be approximated numerically or graphically.



$y = y(x)$: solution curve

slope of the tangent line to the solution curve is $m = \frac{dy}{dx} = f(x, y)$
↑ given/known function

at every pt (x, y) we draw a small line segment whose slope $= f(x, y)$. The collection of all such small segments is called slope field or direction field.

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If a solution curve goes through pt (x,y), the slope of its tangent line at this point is $f(x,y)$ (= slope of small segment).

<u>Ex</u> $\frac{dy}{dx} = ky$, $k=1 \to \frac{dy}{dx} = \underbrace{y}_{f(x,y)} \Rightarrow$ slope $\boxed{m = y} = f(x,y)$

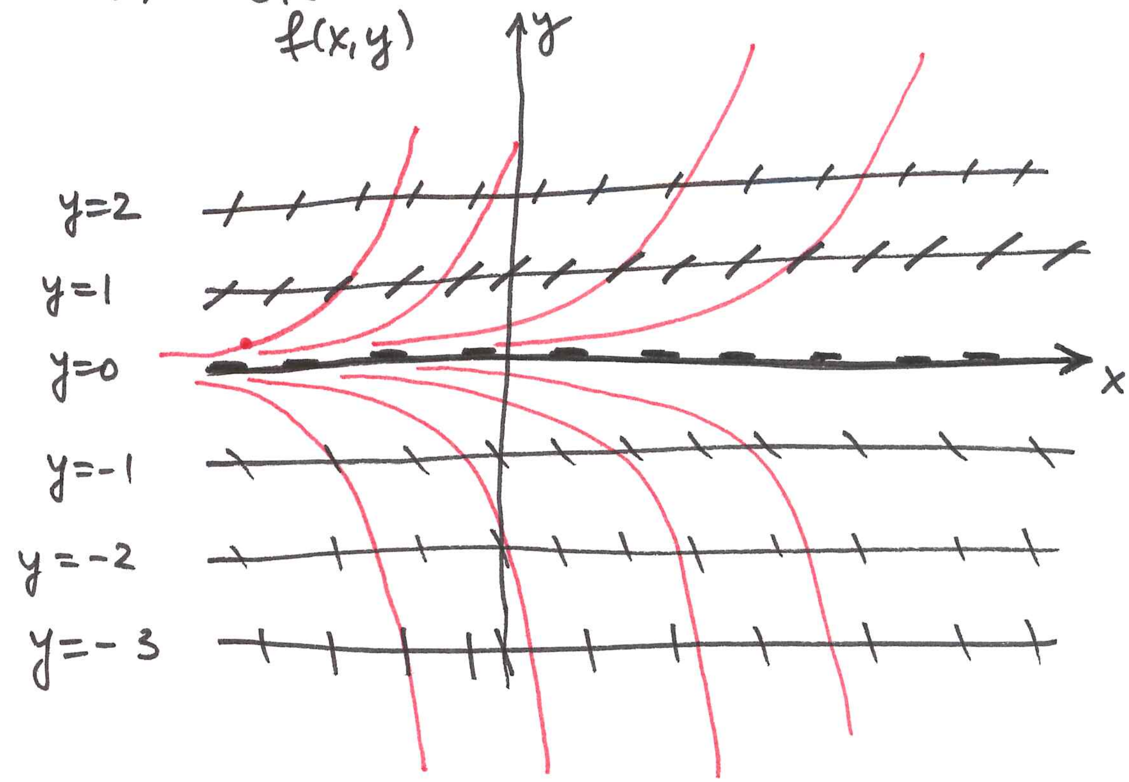
x-axis $\Rightarrow y=0 \Rightarrow m = f(x,y) = 0$

$y=1 \Rightarrow m=1$

$y=2 \Rightarrow m=2$

$y=-1 \Rightarrow m=-1$

$y=-3 \Rightarrow m=-3$



Ex $\dfrac{dy}{dx} = \underbrace{x - y}_{f(x,y)} = m$

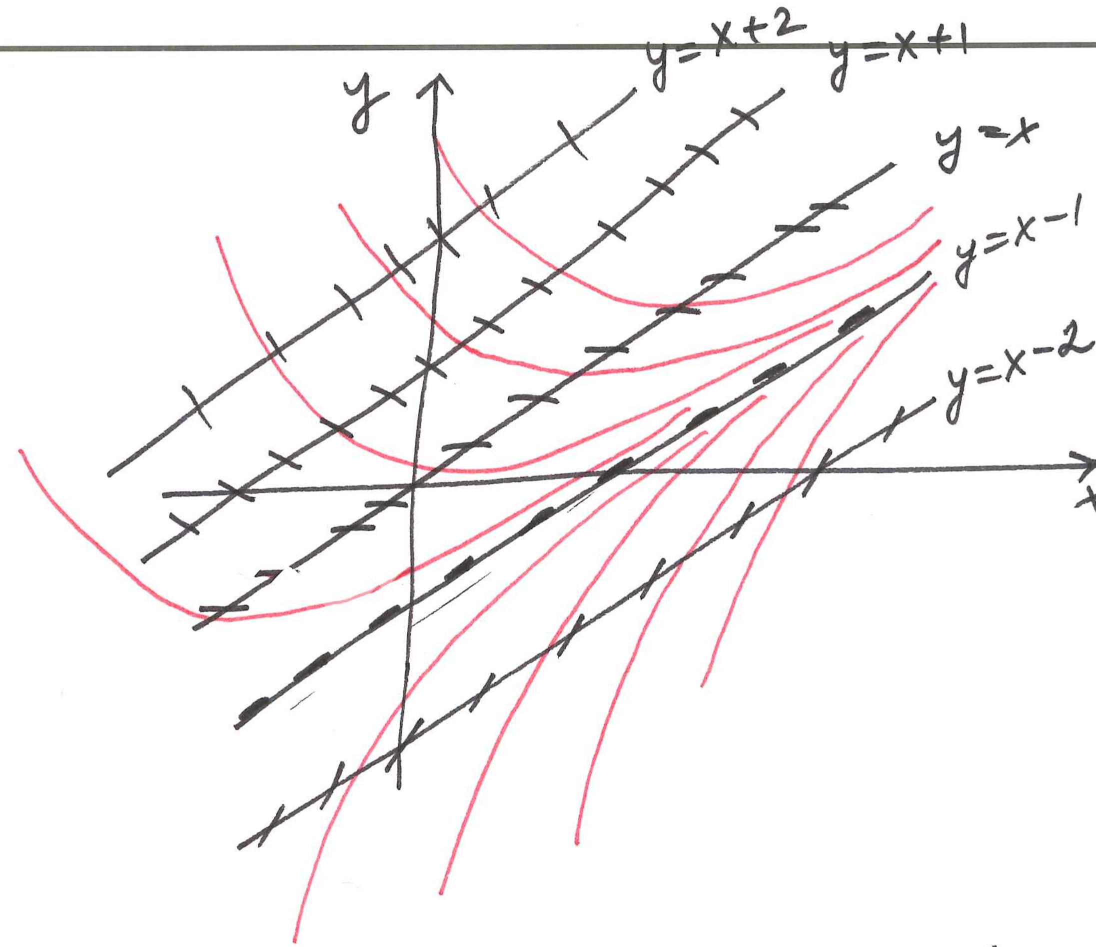
Let $y = x \Rightarrow m = 0$

$y = x+1 \Rightarrow m = x - (x+1) = -1$

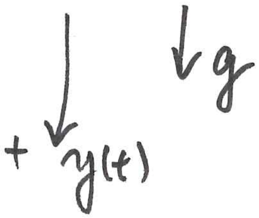
$y = x+2 \Rightarrow m = x - (x+2) = -2$

$y = x-1 \Rightarrow m = x - (x-1) = 1$

$y = x-2 \Rightarrow m = x - (x-2) = 2$



Ex Vertical motion of an object under the action of gravity w/ air resistance.



$y(t)$: position of an object

$v(t) = \dfrac{dy}{dt}$

$$\frac{\text{Force}}{m} = \frac{dv}{dt} = \underset{32}{\underset{\shortparallel}{g}} - kv$$

$k = 0.16$: typical air resistance coefficient

$\Rightarrow$ DE is $\frac{dv}{dt} = g - kv$ or $\frac{dv}{dt} = 32 - 0.16v$

Equilibrium solution is a solution for which total Force $= 0$

$\Rightarrow \frac{dv}{dt} = 0$: all forces are in balance

$32 - 0.16v = 0 \Rightarrow v = \frac{32}{0.16} = 200$ ft/s : <u>equilibrium velocity</u> or <u>limiting velocity</u>