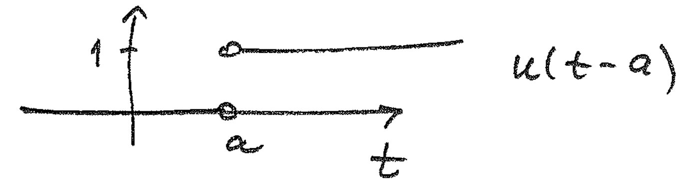
Lecture 40

(Section 7.5)

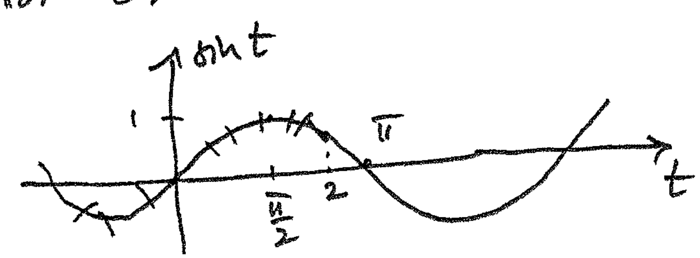
Recall

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

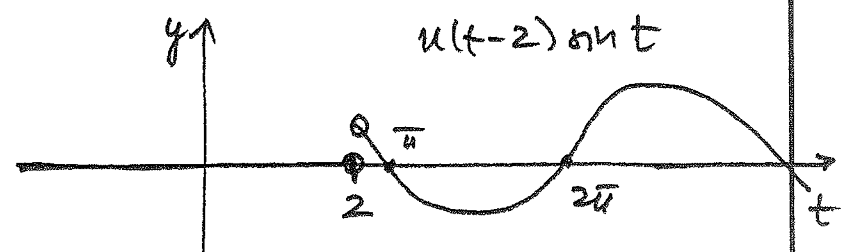


Graph function $u(t-2)\sin t$.

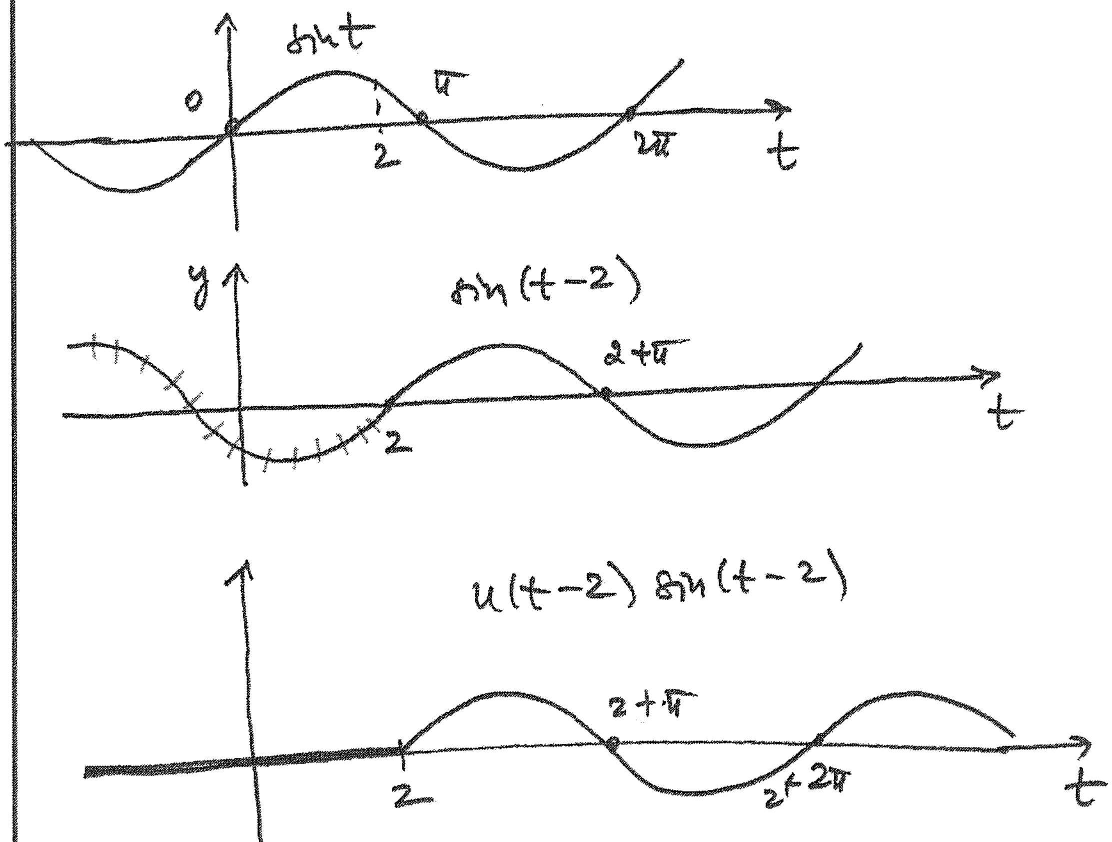
The easiest way is to graph $\sin t$ and then "erase" (make zero or multiply by zero) the graph of $\sin t$ to the left from $t=2$ and leave the graph unaltered (multiply by 1) for $t > 2$.



$\Longrightarrow$



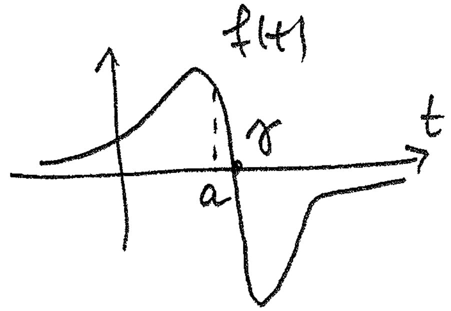
Graph $u(t-2)\sin(t-2)$. $\sin(t-2)$ resembles $\sin t$ but it is shifted by 2 units to the right. Then $u(t-2)$ "erases" the graph for $t<2$.

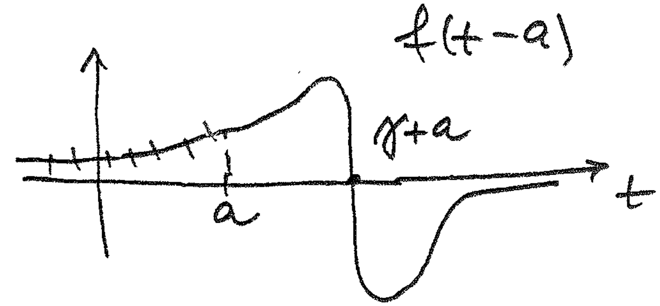


Because of $u(t-2)$, we "chop off" the graph to the left of 2

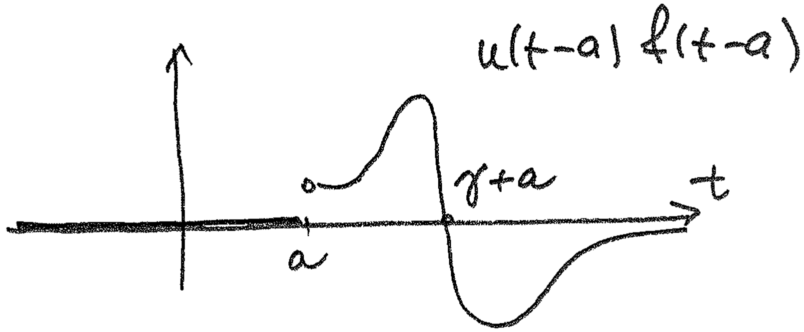
<u>Q</u> What is the graph of $u(t-a)\,f(t-a)$?

<u>A</u> $u(t-a)\,f(t-a)$ looks like $f(t)$ but shifted by a units to the right, i.e. it is $f(t-a)$ for $t>a$, and it is zero for $t<a$





$\Downarrow$



Thm (Shift-Chop Thm)

$$\mathcal{L}\{u(t-a)\, f(t-a)\} = e^{-as} F(s), \quad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

i.e. Laplace transform of shifted and "chopped" function equals Laplace transform of the original function (not shifted) multiplied by e^{-as}.

Problem Graph $\mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^2} \right\}$

$$\mathcal{L}\{u(t-a)\, f(t-a)\} = e^{-as} F(s)$$

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, f(t-a)$$

$e^{-2s} \cdot \frac{1}{s^2}$

$a=2$

$\frac{1}{s^2} = \mathcal{L}\{t\} \Rightarrow f(t) = t$

$t \to t-a = t-2$

$u(t-2)$

Recall

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$\therefore \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s^2}\right\} = (t-2)\cdot u(t-2)$

$(t-2)\cdot u(t-2)$

$a=2$

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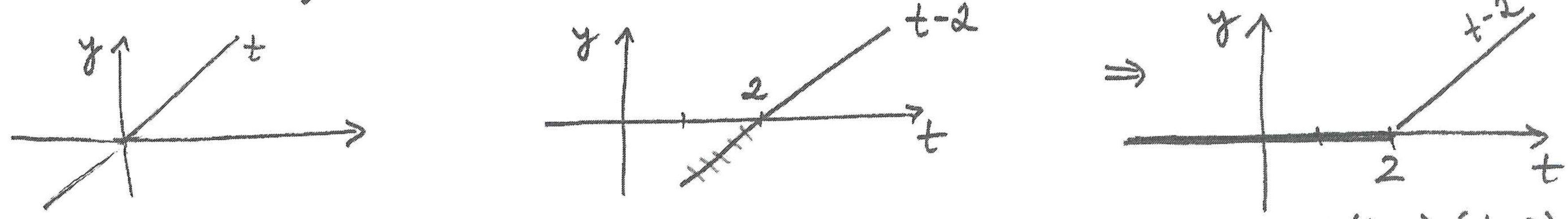
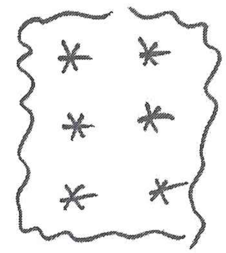


$u(t-2)\,(t-2)$

Ex Find $\mathcal{L}^{-1}\left\{ \dfrac{e^{-s}(2s+1)}{s^2+2s+5} \right\}$



WARNING

УВАГА

DANGER

НЕБЕЗПЕКА

$F(s) = \dfrac{2s+1}{s^2+2s+5} = $

In this case we complete the square:

$s^2+2s+5 = s^2+2s+1+4 = (s+1)^2+2^2$

$= \dfrac{2s+1}{(s+1)^2+2^2} = \dfrac{2(s+1)-2+1}{(s+1)^2+2^2} =$

You can apply partial fraction decomposition to quotient of polynomials only!

$2s+1 = 2(s+1) - 2 + 1$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$a=-1$

$= \dfrac{2(s+1)-1}{(s+1)^2+2^2} = \dfrac{2(s+1)}{(s+1)^2+2^2} - \dfrac{1}{(s+1)^2+2^2}$

$\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$ $\qquad s+1 \Rightarrow e^{-t}$, $a=-1$ $\qquad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\therefore$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right\} =$$

$$2\cos 2t \cdot e^{-t} \qquad - \frac{1}{2}\sin 2t \cdot e^{-t}$$

$\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

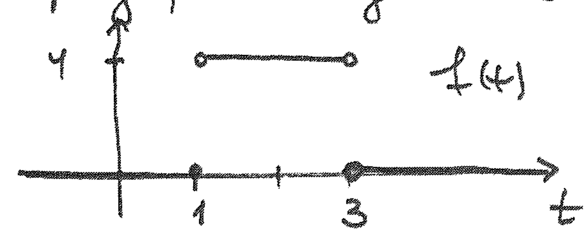
$$= 2\cos 2t \cdot e^{-t} - \frac{1}{2}\sin 2t \cdot e^{-t} = f(t)$$

Recall Shift-Chop Thm : $\mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, f(t-a)$

$$\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{2s+1}{s^2+2s+5}\right\} = \left[2\cos 2(t-1) \cdot e^{-(t-1)} - \frac{1}{2}\sin 2(t-1) \cdot e^{-(t-1)}\right] u(t-1)$$

$a=1 \Rightarrow t \to t-1$, $u(t-1)$

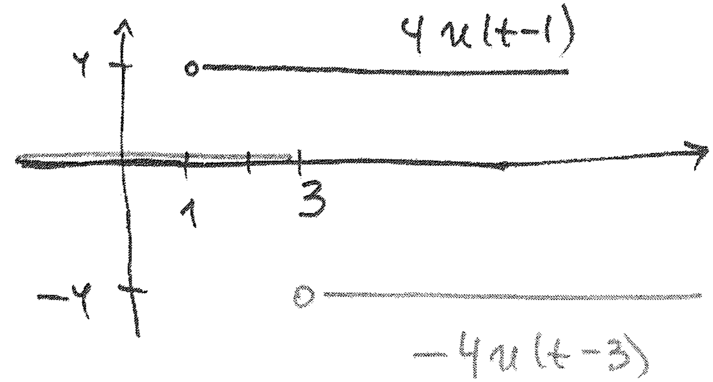
Problem Mass-spring system with damping, driving force $f(t)$, satisfies the ODE



$$\ddot{x} + 3\dot{x} + 2x = f(t)$$

$$x(0) = 2, \quad \dot{x}(0) = -1$$

Note $f(t) = 4u(t-1) - 4u(t-3)$



Apply Laplace transform to both sides of DE.

Recall $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$\Big(s^2 X(s) - s\underset{2}{x(0)} - \underset{-1}{x'(0)}\Big) + 3\Big(s X(s) - \underset{2}{x(0)}\Big) + 2X(s) = 4\frac{e^{-s}}{s} - 4\frac{e^{-3s}}{s}$$

$$s^2 X(s) - 2s + 1 + 3\left(s X(s) - 2\right) + 2 X(s) = 4\frac{e^{-s}}{s} - 4\frac{e^{-3s}}{s}$$

$$\left(s^2+3s+2\right) X(s) = \underbrace{2s - 1 + 6}_{2s+5} + 4\frac{e^{-s}}{s} - 4\frac{e^{-3s}}{s}$$

$$X(s) = \frac{2s+5}{s^2+3s+2} + e^{-s} \cdot \frac{4}{s(s^2+3s+2)} - e^{-3s} \cdot \frac{4}{s(s^2+3s+2)}$$

keep the terms with e^{-s} and e^{-3s} separately, do not mix them!

Ex (Cont'd)

$$X(s) = \frac{2s+5}{s^2+3s+2} + e^{-s}\frac{4}{s(s^2+3s+2)} - e^{-3s}\cdot\frac{4}{s(s^2+3s+2)}$$

$$s^2+3s+2 = (s+1)(s+2)$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\frac{2s+5}{s^2+3s+2} \overset{\text{partial}}{\underset{\text{fraction decomposition}}{=}} \frac{3}{s+1} + \frac{-1}{s+2}$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\frac{4}{s(s^2+3s+2)} = \frac{2}{s} + \frac{-4}{s+1} + \frac{2}{s+2}$$

$4e^{-(t-1)}$ $\quad 2e^{-2(t-1)}$ $\quad 1 \quad 4e^{-(t-3)} \quad 2e^{-2(t-3)}$

$$X(s) = \left(\frac{3}{s+1} - \frac{1}{s+2}\right) + \underset{\substack{a=1 \\ t\to t-1}}{e^{-s}}\left(\frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}\right) - \underset{\substack{a=3 \\ t\to t-3}}{e^{-3s}}\left(\frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}\right)$$

$$x(t) = 3e^{-t} - e^{-2t} + \underbrace{\left[2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right]\cdot u(t-1)}_{=0 \text{ up to } t=1}$$

$$- \underbrace{\left[2 - 4e^{-(t-3)} + 2e^{-2(t-3)}\right]\cdot u(t-3)}_{=0 \text{ up to } t=3}$$

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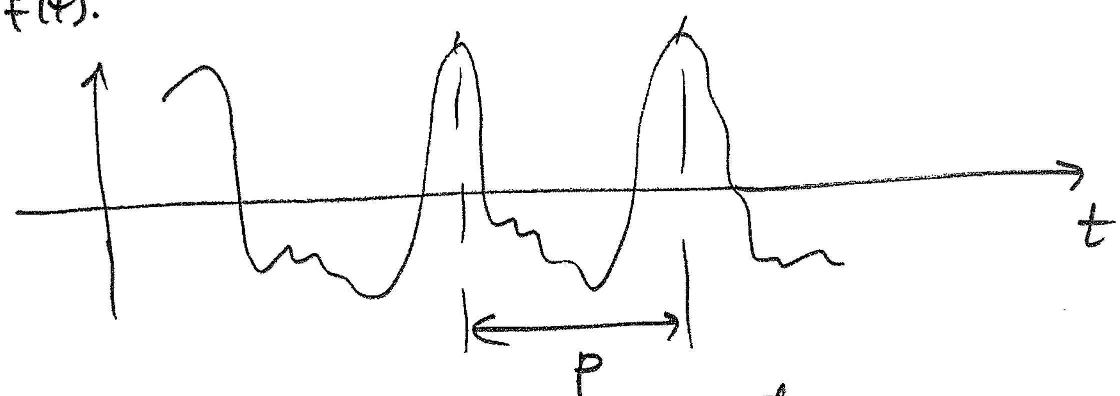
Transforms of Periodic Functions (S 7.5)

Def The nonconstant function $f(t)$, $t \geq 0$, is *periodic* if there exists number $p > 0$:

$$f(t+p) = f(t) \quad \text{for any } t \geq 0 \qquad (1)$$

The least $p > 0$ for which (1) is satisfied, is called *period* of $f(t)$.



Thm Transforms of Periodic Functions

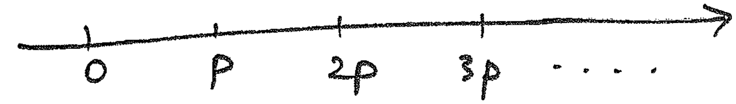
Let $f(t)$ be periodic w/ period p and piecewise continuous.

Then

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

proper integral (defined on a finite interval)

<u>Proof</u>



$$F(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t)\,dt =$$

$$= \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t)\,dt \;\; = \qquad e^{a+b} = e^a \cdot e^b$$

$$\int_{np}^{(n+1)p} e^{-st} f(t)\,dt = \left| \begin{array}{l} t = \tau + np \\ dt = d\tau \\ t = np \Rightarrow \tau = 0 \\ t = (n+1)p \Rightarrow \tau = p \end{array} \right| = \int_0^p \underbrace{e^{-s(\tau+np)}}_{e^{-s\tau} \cdot e^{-snp}} \underbrace{f(\tau + np)}_{f(\tau) \text{ by periodicity}} d\tau =$$

$$= e^{-snp} \int_0^p e^{-s\tau} f(\tau)\,d\tau$$

$$= \left(1 + e^{-sp} + e^{-2sp} + e^{-3sp} + \dots\right) \int_0^p e^{-s\tau} f(\tau)\,d\tau \;\; \boxed{=}$$

Recall

$$1 + x + x^2 + \dots = \frac{1}{1-x} \quad \text{if} \quad |x| < 1$$

geometric series

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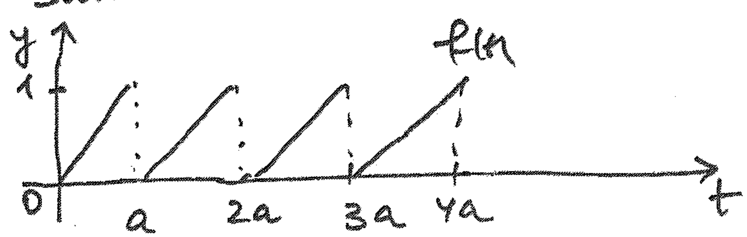
Let $x = e^{-sp} < 1$ for $s > 0$

$$e^{-2sp} = (e^{-sp})^2, \quad e^{-3sp} = (e^{-sp})^3 + \dots$$

$$\boxed{=} \frac{1}{1-e^{-sp}} \int_0^p e^{-s\tau} f(\tau)\, d\tau$$

Ex
#26
S7.5

Sawtooth function



periodic with $p = a$

$$f(t) = \frac{t}{a} \quad \text{on} \quad t \in [0, a)$$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{1-e^{-as}} \int_0^a e^{-s\tau} \cdot \frac{\tau}{a}\, d\tau \underset{\text{by parts}}{\overset{\text{integration}}{=}}$$

$$\left| \begin{array}{ll} u = \tau & dv = e^{-s\tau} d\tau \\ du = d\tau & v = -\frac{e^{-s\tau}}{s} \end{array} \right|$$

$$= \frac{1}{a(1-e^{-as})} \left[-\frac{\tau e^{-s\tau}}{s} \Big|_{\tau=0}^{\tau=a} + \frac{1}{s} \underbrace{\int_0^a e^{-s\tau}\, d\tau}_{= -\frac{1}{s} e^{-s\tau} \big|_{\tau=0}^{\tau=a}} \right] =$$

$$= \frac{1}{a(1-e^{-as})} \left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}(e^{-as} - 1) \right] = -\frac{1}{s} \frac{e^{-as}}{1-e^{-as}} + \frac{1}{as^2}$$

7.6 Impulses and Delta Functions

Consider a force acting over a very short interval of time

Ex Impulsive force of a bat striking a ball or quick surge of voltage from a lighting bolt.

In these case we may only need to know

$$p = \int_a^b f(t)\,dt \; : \; \underline{\text{impulse}} \text{ of } f(t) \text{ over } [a, b]$$

Ex Particle of mass m with linear motion

Newton's 2^{nd} law:

$$f(t) = m \cdot v'(t) = \frac{d}{dt}(m\,v(t))$$

Impulse

$$p = \int_a^b f(t)\,dt = \int_a^b \frac{d}{dt}(m v(t))\,dt = m v(b) - m v(a)$$

$\therefore$ impulse of force $=$ change of momentum of particle

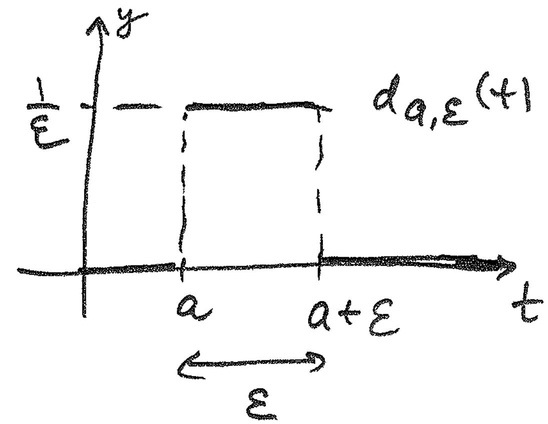
Hence, if we need to know only change of momentum, we need to know only impulse of function and not precise f(t) or even precise time interval

We would like to replace such a force f(t) that acts on a very small time interval w/ a simple model that has the same impulse.

For simplicity, let $p=1$.

Replace $f(t)$ with

$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$



ε: small

Impulse of $d_{a,\varepsilon}(t)$:

$$p = \int_a^b d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1 \quad \checkmark$$

$b = a+\varepsilon$

<u>Note</u> : $d_{a,\varepsilon}(t)$ has the same impulse for any $\varepsilon > 0 \Rightarrow$ we can write

$$\int_0^\infty d_{a,\varepsilon}(t)\,dt = 1$$

Can we think of <u>instantaneous impulse</u> at $t=a$?

Define

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t), \quad a \geq 0$$

<u>DIRAC delta function</u>

Then

$$\int_0^\infty \delta_a(t)\,dt = \int_0^\infty \lim_{\varepsilon\to 0} d_{a,\varepsilon}(t)\,dt \underset{\substack{\int \text{ and } \lim \\ \text{if we can}}}{\overset{\text{swap}}{=}} \lim_{\varepsilon\to 0} \int_0^\infty d_{a,\varepsilon}(t)\,dt = \lim_{\varepsilon \to 0} 1 = 1$$

$$\Rightarrow \boxed{\int_0^\infty \delta_a(t)\,dt = 1} \qquad (1)$$

But

$$\boxed{\delta_a(t) = \begin{cases} +\infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}} \qquad (2)$$

No actual function can satisfy both (1) and (2)

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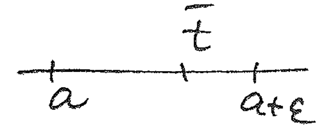
Delta functions as operators

If $g(t)$ is a continuous function, then by Mean Value Thm

$$\int_a^{a+\varepsilon} g(t)\,dt = \varepsilon\, g(\bar{t}) \qquad \text{for some} \quad a \le \bar{t} \le a+\varepsilon$$



Then

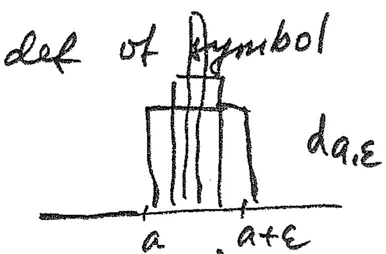
$$\lim_{\varepsilon\to 0} \int_0^\infty g(t)\, d_{a,\varepsilon}(t)\,dt = \lim_{\varepsilon\to 0} \int_a^{a+\varepsilon} g(t)\cdot\frac{1}{\varepsilon}\,dt = \lim_{\varepsilon\to 0} g(\bar{t}) = g(a)$$

If $\delta_a(t)$ were a function in the strict sense of def, and if we could interchange lim and $\int$, we could get

$$\int_0^\infty g(t)\,\delta_a(t)\,dt = \int_0^\infty g(t) \lim_{\varepsilon\to 0} d_{a,\varepsilon}(t)\,dt \underset{\substack{\text{swap}\\ \int \text{and lim}}}{\overset{?}{=}} \lim_{\varepsilon\to 0} \int_0^\infty g(t)\cdot d_{a,\varepsilon}(t)\,dt = g(a)$$

$$\Rightarrow \boxed{\int_0^\infty g(t)\,\delta_a(t)\,dt = g(a)}$$

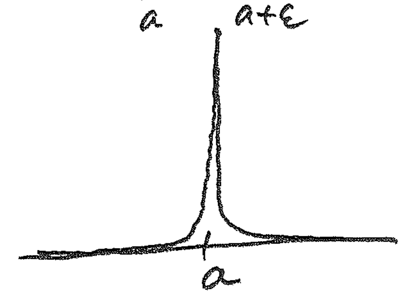
We take this as def of symbol $\delta_a(t)$



$\delta_a(t)$ specifies the operation $\int_0^\infty \ldots \delta_a(t)\,dt$

generalized function, an operator

$$g(t) : \xrightarrow{\delta_a(t)} g(a) \qquad \text{operator}$$



Ex $g(t) = e^{-st}$

$$\int_0^\infty e^{-st}\,\delta_a(t)\,dt = e^{-as} \quad \Rightarrow \quad \boxed{\mathcal{L}\{\delta_a(t)\} = e^{-as}, \quad s \geq 0}$$

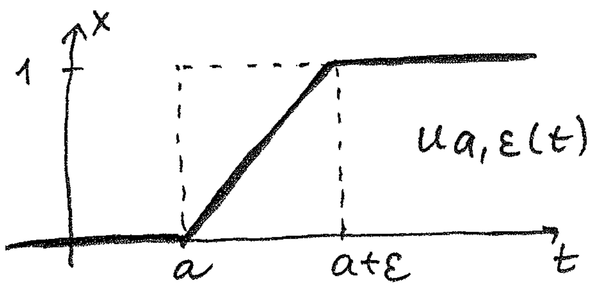
Notation $\delta(t) = \delta_0(t)$, $\delta(t-a) = \delta_a(t)$

Let $a=0$ $\Rightarrow$ $\boxed{\mathcal{L}\{\delta(t)\} = 1}$ $\not\to 0$ as $s \to \infty$

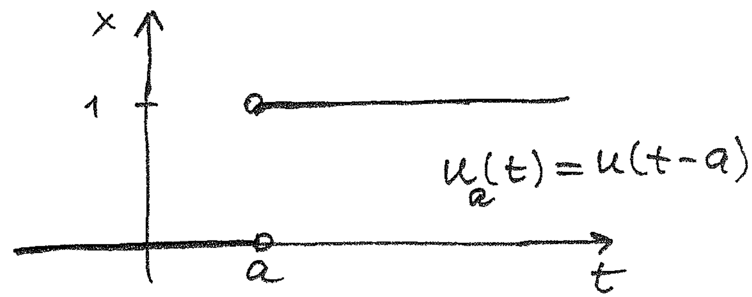
(but $\delta(t)$ is not an ordinary function)

Delta Functions and Step Functions

Delta function $\delta_a(t)$ can be regarded as the derivative of unit step function $u_a(t)$.

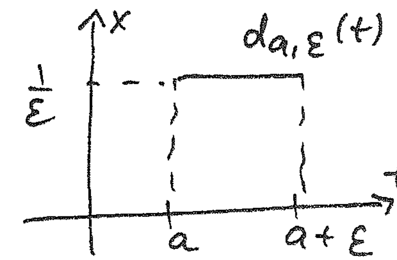


approximation of $u(t-a)$



$$\lim_{\varepsilon \to 0} u_{a,\varepsilon}(t) = u_a(t)$$

$$\frac{d}{dt} u_{a,\varepsilon}(t) = \left\{ \begin{array}{ll} 0, & t \le a \\ \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \\ 0, & t > a+\varepsilon \end{array} \right\} = d_{a,\varepsilon}(t)$$



Recall $\quad \delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t)$

Hence,

$$\frac{d}{dt} u_a(t) = \frac{d}{dt} \lim_{\varepsilon \to 0} u_{a,\varepsilon}(t) \underset{\frac{d}{dt} \text{ and } \lim}{\overset{\text{swap}}{=}} \lim_{\varepsilon \to 0} \frac{d}{dt} u_{a,\varepsilon}(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \delta_a(t)$$

$$\boxed{\frac{d}{dt} u_a(t) = \delta_a(t) = \delta(t-a)}$$

formal def of derivative of unit step function

<u>Note</u> $u_a(t)$ is not differentiable in the ordinary sense at $t=a$

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#4 S7.6

Solve

$$x'' + 2x' + x = t + \delta(t), \qquad x(0) = 0, \quad x'(0) = 1$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$\mathcal{L}\{\delta(t-0)\} = 1, \quad \delta(t-0) = \delta(t)$$

Apply Laplace transform method. Let $X(s) = \mathcal{L}\{x(t)\}$.

$$\Big(s^2 X(s) - s \underbrace{x(0)}_{0} - \underbrace{x'(0)}_{1}\Big) + 2\Big(s X(s) - \underbrace{x(0)}_{0}\Big) + X(s) = \frac{1}{s^2} + 1$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$(s^2 + 2s + 1) X(s) = \frac{1}{s^2} + 2$$

$$X(s) = \frac{\frac{1}{s^2} + 2}{s^2 + 2s + 1} = \frac{2s^2 + 1}{s^2(s^2 + 2s + 1)} = \frac{2s^2 + 1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

partial fraction decomposition

$A = -2$, $B = 1$, $C = 2$, $D = 3$

$$X(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\boxed{x(t) = \mathcal{L}^{-1}\{X(s)\} = -2 + t + 2e^{-t} + 3te^{-t}}$$

#18
S7.6

Consider an initially passive LC circuit (no resistance) with a battery supplying e_0 volts.

(a) If the switch is closed at time $t=0$ and opened at time $t=a>0$, show that the current in the circuit satisfies the IVP

$$Li'' + \frac{1}{C} i = e_0 \delta(t) - e_0 \delta(t-a)$$

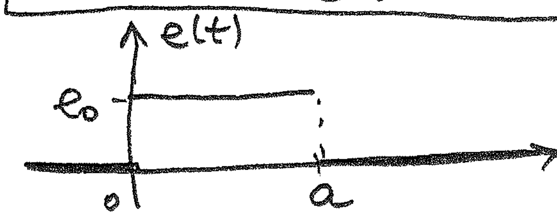
$$i(0) = i'(0) = 0$$

$i = i(t)$: current
$q = q(t)$: charge
$I(s) = \mathcal{L}\{i(t)\}$

(b) If $L = 1\,H$, $C = 10^{-2}\,F$, $e_0 = 10\,V$ and $a = \pi\,(s)$, show that

$$i(t) = \begin{cases} \sin 10t, & t < \pi \\ 0, & t > \pi \end{cases}$$

(*) $\boxed{Li' + Ri + \frac{1}{C} q = e(t)}$



Initially passive circuit: $i(0) = 0$, $q(0) = 0$.

No resistance $\Rightarrow$ $R = 0$.

$$Li' + \frac{1}{C} q = e_0 [u(t) - u(t-a)]$$

$$e(t) = e_0 u(t) - e_0 u(t-a) = e_0 [u(t) - u(t-a)]$$

Differentiate wrt t. $\qquad \frac{d}{dt} q(t) = i(t)$

$$L i'' + \frac{1}{C} i = e_0 [\delta(t) - \delta(t-a)]$$

From (*), we can find $i'(0) = \dfrac{\overset{0}{e(0)} - \frac{1}{C}\overset{0}{q(0)}}{L} = 0 \Rightarrow \boxed{i'(0)=0}$

(b) $L=1$, $C=10^{-2}$, $e_0=10$, $a=\pi$ $\qquad \frac{1}{10^{-2}} = 100$

$$i'' + 100 i = 10[\delta(t) - \delta(t-\pi)], \quad i(0)=0, \quad i'(0)=0$$

$\mathcal{L}\{\delta(t-a)\} = e^{-as}$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$$\left(s^2 I(s) - s\cancel{i(0)}^{0} - \cancel{i'(0)}^{0}\right) + 100\, I(s) = 10\left[1 - e^{-\pi s}\right]$$

$$(s^2+100) I(s) = 10[1 - e^{-\pi s}]$$

$\mathcal{L}\{u(t-a) f(t-a)\} = e^{-as} F(s)$

$$I(s) = \frac{10}{s^2+100} - e^{-\pi s} \cdot \frac{10}{s^2+100}$$

$$i(t) = \sin 10t - \underbrace{\sin 10(t-\pi)}_{\sin(10t - 10\pi) = \sin 10t} \cdot u(t-\pi) = \sin 10t\,[1 - u(t-\pi)]$$

periodic

$f(t+p) = f(t)$

$f(t \pm np) = f(t)$

$$= \sin 10t - \sin 10t \cdot u(t-\pi) = \sin 10t\,[1 - u(t-\pi)]$$

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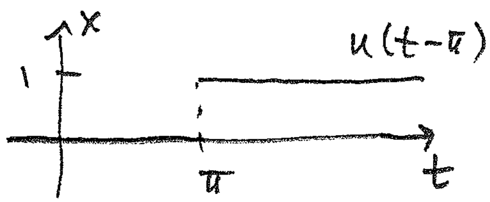

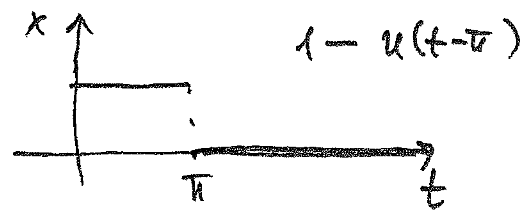

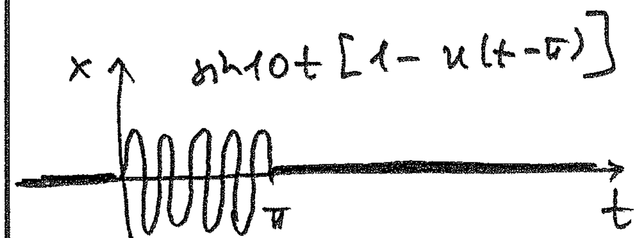

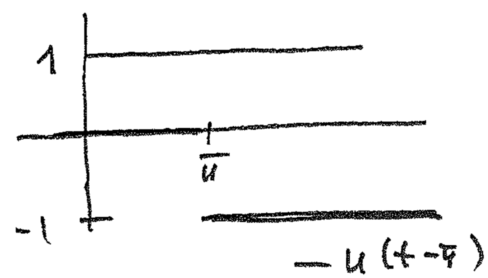

$$\Rightarrow \quad x(t) = \begin{cases} \sin 10t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$$