University of Idaho

Lecture 40

Ex (Cont'd)

$$X(s) = \frac{2s+5}{s^2+3s+2} + e^{-s}\frac{4}{s(s^2+3s+2)} - e^{-3s}\cdot\frac{4}{s(s^2+3s+2)}$$

$$s^2+3s+2 = (s+1)(s+2)$$

$$\frac{2s+5}{s^2+3s+2} \overset{\text{partial fraction decomposition}}{=} \frac{3}{s+1} + \frac{-1}{s+2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\frac{4}{s(s^2+3s+2)} = \frac{2}{s} + \frac{-4}{s+1} + \frac{2}{s+2}$$

$$X(s) = \left(\frac{3}{s+1} - \frac{1}{s+2}\right) + \underset{\substack{a=1\\ t\to t-1}}{e^{-s}}\left(\frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}\right) - \underset{\substack{a=3\\ t\to t-3}}{e^{-3s}}\left(\frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}\right)$$

$$x(t) = 3e^{-t} - e^{-2t} + \underbrace{\left[2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right]\cdot u(t-1)}_{=0 \text{ up to } t=1}$$

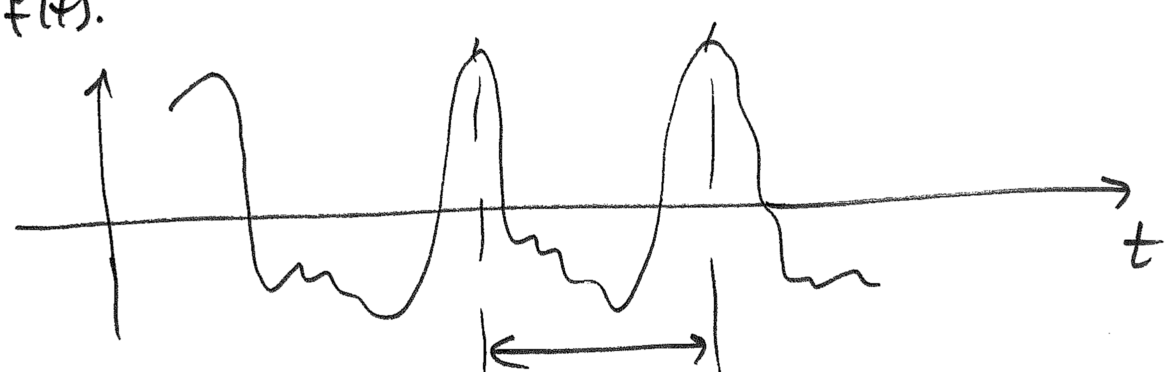
$$- \underbrace{\left[2 - 4e^{-(t-3)} + 2e^{-2(t-3)}\right]\cdot u(t-3)}_{=0 \text{ up to } t=3}$$

Transforms of Periodic Functions (S 7.5)

Def The nonconstant function $f(t)$, $t \geq 0$, is *periodic* if there exists number $p > 0$:

$$f(t+p) = f(t) \quad \text{for any } t \geq 0 \qquad (1)$$

The least $p > 0$ for which (1) is satisfied, is called *period* of $f(t)$.



Thm Transforms of Periodic Functions

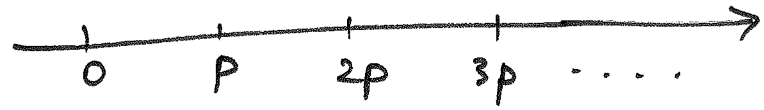
Let $f(t)$ be periodic w/ period p and piecewise continuous.

Then

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

proper integral (defined on a finite interval)

Proof



$$F(s) \overset{def}{=} \int_0^\infty e^{-st} f(t)\, dt =$$

$$= \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t)\, dt \quad (=)$$

$$e^{a+b} = e^a \cdot e^b$$

$$\int_{np}^{(n+1)p} e^{-st} f(t)\, dt = \left| \begin{array}{l} t = \tau + np \\ dt = d\tau \\ t = np \Rightarrow \tau = 0 \\ t = (n+1)p \Rightarrow \tau = p \end{array} \right| = \int_0^p \underbrace{e^{-s(\tau+np)}}_{e^{-s\tau} \cdot e^{-snp}} \underbrace{f(\tau+np)}_{f(\tau) \text{ by periodicity}} d\tau =$$

$$= e^{-snp} \int_0^p e^{-s\tau} f(\tau)\, d\tau$$

$$(=) \left(1 + e^{-sp} + e^{-2sp} + e^{-3sp} + \dots \right) \int_0^p e^{-s\tau} f(\tau)\, d\tau \quad \boxed{\equiv}$$

Recall

$$1 + x + x^2 + \dots = \frac{1}{1-x} \quad \text{if } |x| < 1$$

geometric series

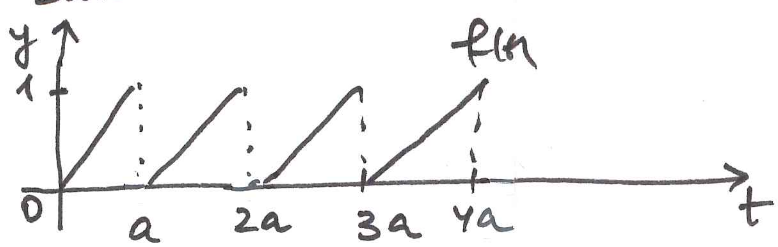
Let $x = e^{-sp} < 1$ for $s > 0$

$$e^{-2sp} = (e^{-sp})^2, \quad e^{-3sp} = (e^{-sp})^3 + \dots$$

$$\equiv \frac{1}{1-e^{-sp}} \int_0^p e^{-s\tau} f(\tau)\, d\tau \qquad \blacksquare$$

Ex #26 S7.5

Sawtooth function



periodic with $p = a$

$$f(t) = \frac{t}{a} \quad \text{on} \quad t \in [0, a)$$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{1-e^{-as}} \int_0^a e^{-s\tau} \cdot \frac{\tau}{a}\, d\tau \underset{\text{by parts}}{\overset{\text{integration}}{=}}$$

$$\begin{vmatrix} u = \tau & dv = e^{-s\tau} d\tau \\ du = d\tau & v = -\frac{e^{-s\tau}}{s} \end{vmatrix}$$

$$= \frac{1}{a(1-e^{-as})}\left[-\frac{\tau e^{-s\tau}}{s}\Big|_{\tau=0}^{\tau=a} + \frac{1}{s}\underbrace{\int_0^a e^{-s\tau}\, d\tau}_{= -\frac{1}{s} e^{-s\tau}\big|_{\tau=0}^{\tau=a}} \right] =$$

$$= \frac{1}{a(1-e^{-as})}\left[-a\frac{e^{-as}}{s} - \frac{1}{s^2}(e^{-as} - 1)\right] = -\frac{1}{s}\frac{e^{-as}}{1-e^{-as}} + \frac{1}{as^2}$$

7.6 Impulses and Delta Functions

Consider a force acting over a very short interval of time

Ex Impulsive force of a bat striking a ball or quick surge of voltage from a lighting bolt.

In these case we may only need to know

$$p = \int_a^b f(t)\,dt \; : \; \underline{\text{impulse}} \text{ of } f(t) \text{ over } [a, b]$$

Ex Particle of mass m with linear motion

Newton's 2^{nd} law:

$$f(t) = m \cdot v'(t) = \frac{d}{dt}\left(m\,v(t)\right)$$

Impulse

$$p = \int_a^b f(t)\,dt = \int_a^b \frac{d}{dt}\left(m v(t)\right) dt = m v(b) - m v(a)$$

Hence, if we need to know only change of momentum, we need to know only impulse of function and not precise f(t) or even precise time interval

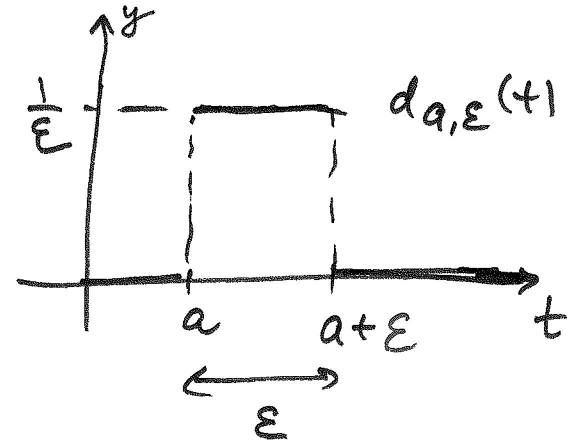
$\therefore$ impulse of force = change of momentum of particle

We would like to replace such a force f(t) that acts on a very small time interval w/ a simple model that has the same impulse.

For simplicity, let $p=1$.

Replace $f(t)$ with

$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$



ε: small

Impulse of $d_{a,\varepsilon}(t)$:

$$p = \int_a^b d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1 \quad \checkmark$$

$b = a+\varepsilon$