Note : $d_{a,\varepsilon}(t)$ has the same impulse for any $\varepsilon > 0$ $\Rightarrow$ we can write

$$\int_0^\infty d_{a,\varepsilon}(t)\,dt = 1$$

can we think of instantaneous impulse at $t=a$?

Define

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t), \quad a \geq 0$$

DIRAC delta function

Then

$$\int_0^\infty \delta_a(t)\,dt = \int_0^\infty \lim_{\varepsilon\to 0} d_{a,\varepsilon}(t)\,dt \underset{\substack{\int \text{and } \lim \\ \text{if we can}}}{\overset{\text{swap}}{=}} \lim_{\varepsilon\to 0}\int_0^\infty d_{a,\varepsilon}(t)\,dt = \lim_{\varepsilon \to 0} 1 = 1$$

$$\Rightarrow \boxed{\int_0^\infty \delta_a(t)\,dt = 1} \qquad (1)$$

but

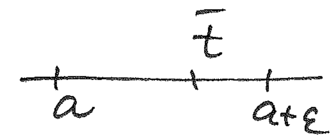
$$\boxed{\delta_a(t) = \begin{cases} +\infty & \text{if } t=a \\ 0 & \text{if } t \neq a \end{cases}} \qquad (2)$$

No actual function can satisfy both (1) and (2)

Delta functions as operators

If $g(t)$ is a continuous function, then by Mean Value Thm

$$\int_a^{a+\varepsilon} g(t)\,dt = \varepsilon\, g(\bar{t}) \qquad \text{for some} \quad a \le \bar{t} \le a+\varepsilon$$



Then

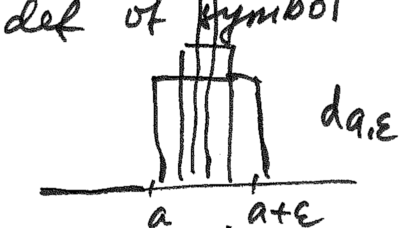
$$\lim_{\varepsilon\to 0} \int_0^\infty g(t)\, d_{a,\varepsilon}(t)\,dt = \lim_{\varepsilon\to 0} \int_a^{a+\varepsilon} g(t)\cdot \frac{1}{\varepsilon}\,dt = \lim_{\varepsilon\to 0} g(\bar{t}) = g(a)$$

If $\delta_a(t)$ were a function in the strict sense of def, and if we could interchange lim and $\int$, we could get

$$\int_0^\infty g(t)\,\delta_a(t)\,dt = \int_0^\infty g(t) \lim_{\varepsilon\to 0} d_{a,\varepsilon}(t)\,dt \underset{\substack{\text{swap}\\ \int \text{ and lim}}}{\overset{?}{=}} \lim_{\varepsilon\to 0} \int_0^\infty g(t)\cdot d_{a,\varepsilon}(t)\,dt = g(a)$$

$$\Rightarrow \boxed{\int_0^\infty g(t)\,\delta_a(t)\,dt = g(a)}$$

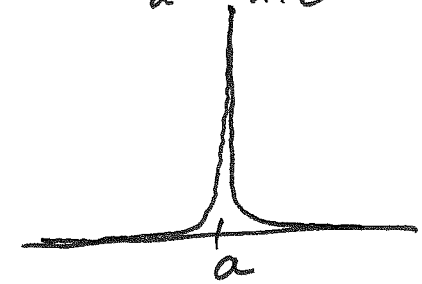
We take this as def of symbol $\delta_a(t)$



$\delta_a(t)$ specifies the operation $\int_0^\infty \dots \delta_a(t)\,dt$

generalized function, an operator

$$g(t) : \xrightarrow{\delta_a(t)} g(a) \qquad \text{operator}$$



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<u>Ex</u> $g(t) = e^{-st}$

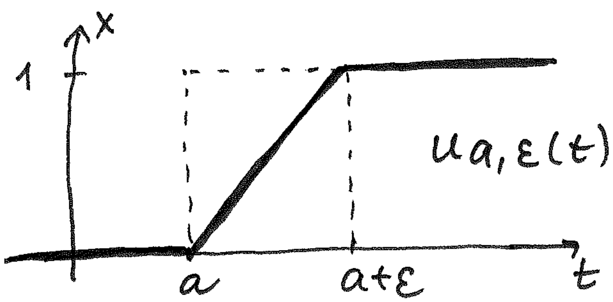
$$\int_0^\infty e^{-st}\,\delta_a(t)\,dt = e^{-as} \quad \Rightarrow \quad \boxed{\mathcal{L}\{\delta_a(t)\} = e^{-as}, \quad s \geq 0}$$

<u>Notation</u> $\delta(t) = \delta_0(t)$, $\delta(t-a) = \delta_a(t)$

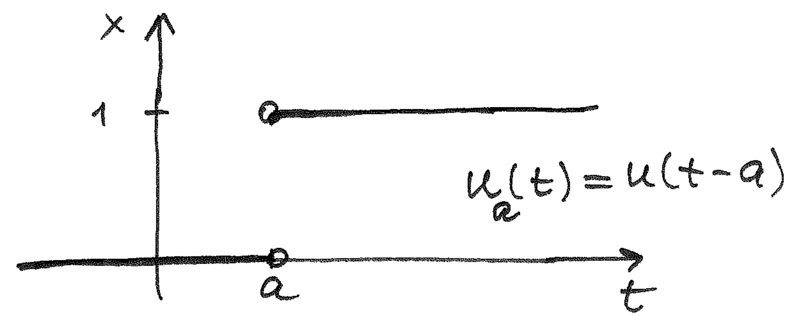
Let $a=0$ $\Rightarrow$ $\boxed{\mathcal{L}\{\delta(t)\} = 1}$ $\not\to 0$ as $s \to \infty$
(but $\delta(t)$ is not an ordinary function)

<u>Delta Functions and Step Functions</u>

Delta function $\delta_a(t)$ can be regarded as the derivative of unit step function $u_a(t)$.

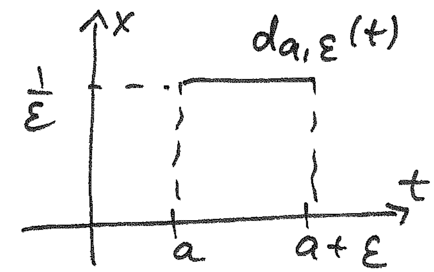


approximation of $u(t-a)$



$$\lim_{\varepsilon \to 0} u_{a,\varepsilon}(t) = u_a(t)$$

$$\frac{d}{dt} u_{a,\varepsilon}(t) = \left\{ \begin{array}{ll} 0, & t \le a \\ \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \\ 0, & t > a+\varepsilon \end{array} \right\} = d_{a,\varepsilon}(t)$$



Recall $\delta_a(t) = \lim\limits_{\varepsilon \to 0} d_{a,\varepsilon}(t)$

Hence,

$$\frac{d}{dt} u_a(t) = \frac{d}{dt} \lim_{\varepsilon \to 0} u_{a,\varepsilon}(t) \underset{\frac{d}{dt} \text{ and } \lim}{\overset{\text{swap}}{=}} \lim_{\varepsilon \to 0} \frac{d}{dt} u_{a,\varepsilon}(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \delta_a(t)$$

$$\boxed{\frac{d}{dt} u_a(t) = \delta_a(t) = \delta(t-a)}$$

formal def of derivative of unit step function

Note $u_a(t)$ is not differentiable in the ordinary sense at $t=a$

#4
S7.6

Solve

$$x'' + 2x' + x = t + \delta(t), \qquad x(0) = 0, \quad x'(0) = 1$$

Apply Laplace transform method. Let $X(s) = \mathcal{L}\{x(t)\}$.

$$\Big(s^2 X(s) - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{1}\Big) + 2\Big(s X(s) - \underbrace{x(0)}_{0}\Big) + X(s) = \frac{1}{s^2} + 1$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$(s^2+2s+1)\, X(s) = \frac{1}{s^2} + 2$$

$$X(s) = \frac{\frac{1}{s^2}+2}{s^2+2s+1} = \frac{2s^2+1}{s^2(s^2+2s+1)} = \frac{2s^2+1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

partial fraction decomposition

$A = -2, \quad B = 1, \quad C = 2, \quad D = 3$

$$X(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\boxed{x(t) = \mathcal{L}^{-1}\{X(s)\} = -2 + t + 2e^{-t} + 3te^{-t}}$$

#18
S7.6

Consider an initially passive LC circuit (no resistance) with a battery supplying e_0 volts.

(a) If the switch is closed at time $t=0$ and opened at time $t=a>0$, show that the current in the circuit satisfies the IVP

$$Li'' + \frac{1}{C} i = e_0 \delta(t) - e_0 \delta(t-a)$$

$$i(0) = i'(0) = 0$$

$i = i(t)$: current

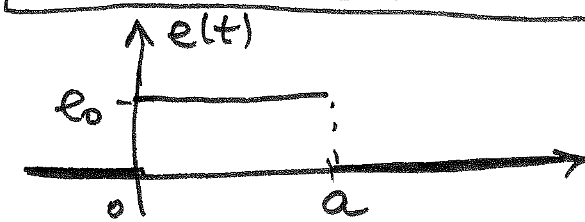
$q = q(t)$: charge

$I(s) = \mathcal{L}\{i(t)\}$

(b) If $L = 1\,H$, $C = 10^{-2}\,F$, $e_0 = 10\,V$ and $a = \pi\,(s)$, show that

$$i(t) = \begin{cases} \sin 10t, & t < \pi \\ 0, & t > \pi \end{cases}$$

(*) $\boxed{Li' + Ri + \frac{1}{C} q = e(t)}$



Initially passive circuit: $i(0) = 0$, $q(0) = 0$.

No resistance $\Rightarrow$ $R = 0$.

$$e(t) = e_0 u(t) - e_0 u(t-a)$$
$$= e_0 [u(t) - u(t-a)]$$

$$Li' + \frac{1}{C} q = e_0 [u(t) - u(t-a)]$$

Differentiate wrt t. $\qquad \frac{d}{dt} q(t) = i(t)$

$$L i'' + \frac{1}{C} i = e_0 [\delta(t) - \delta(t-a)]$$

From (*), we can find $i'(0) = \dfrac{\overset{0}{e(0)} - \frac{1}{C} \overset{0}{q(0)}}{L} = 0 \Rightarrow \boxed{i'(0) = 0}$

(b) $L = 1$, $C = 10^{-2}$, $e_0 = 10$, $a = \pi$ $\qquad \frac{1}{10^{-2}} = 100$

$$i'' + 100 i = 10[\delta(t) - \delta(t-\pi)], \quad i(0) = 0, \quad i'(0) = 0$$

$\mathcal{L}\{\delta(t-a)\} = e^{-as}$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$$\left(s^2 I(s) - s\cancel{i(0)}^{0} - \cancel{i'(0)}^{0}\right) + 100 I(s) = 10[1 - e^{-\pi s}]$$

$$(s^2 + 100) I(s) = 10[1 - e^{-\pi s}]$$

$\mathcal{L}\{u(t-a) f(t-a)\} = e^{-as} F(s)$

$$I(s) = \frac{10}{s^2+100} - e^{-\pi s} \cdot \frac{10}{s^2+100}$$

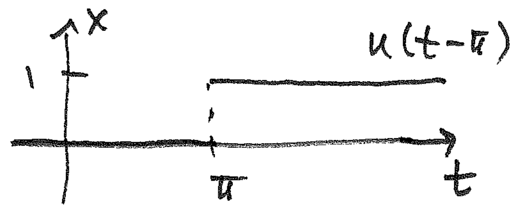
$$i(t) = \sin 10t - \underbrace{\sin 10(t-\pi)}_{\sin(10t - 10\pi) = \sin 10t} \cdot u(t-\pi) = \sin 10t [1 - u(t-\pi)]$$

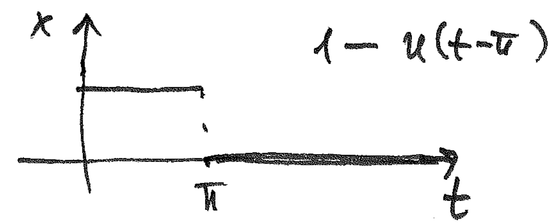
periodic

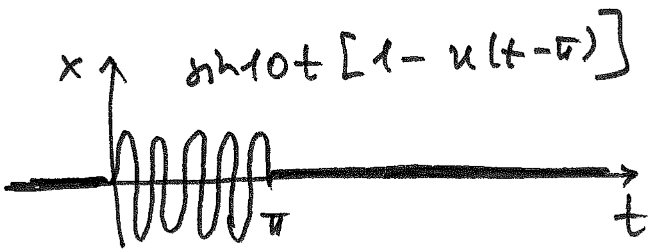
$f(t+p) = f(t)$

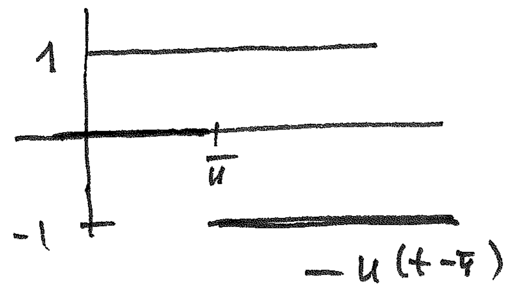
$f(t \pm np) = f(t)$

$$= \sin 10t - \sin 10t \cdot u(t-\pi) = \sin 10t [1 - u(t-\pi)]$$









$$\Rightarrow x(t) = \begin{cases} \sin 10t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$$

#3 S7.4

Compute

$$\sin t * \sin t = \int_0^t \sin\tau \cdot \sin(t-\tau)\,d\tau = \int_0^t \sin\tau\,(\sin t\cdot\cos\tau - \cos t\cdot\sin\tau)\,d\tau \;\text{(=)}$$

(1) $\sin(t-\tau) = \sin t\cdot\cos\tau - \cos t\cdot\sin\tau$

$$\text{(=)} \int_0^t \left(\sin t\cdot \sin\tau\cos\tau - \cos t\cdot\sin^2\tau\right)d\tau = \int_0^t \sin t\cdot\tfrac{1}{2}\sin 2\tau\,d\tau - \int_0^t \cos t\cdot\frac{1-\cos 2\tau}{2}\,d\tau \;\text{(=)}$$

$\boxed{\sin\tau\cdot\cos\tau = \frac{1}{2}\sin 2\tau}$ $\boxed{\sin^2\tau = \frac{1-\cos 2\tau}{2}}$ $\boxed{\cos^2\tau = \frac{1+\cos 2\tau}{2}}$

$$\text{(=)}\ \frac{1}{2}\sin t\cdot\int_0^t \sin 2\tau\,d\tau - \frac{1}{2}\cos t\int_0^t (1-\cos 2\tau)\,d\tau = \frac{1}{2}\sin t\cdot\left(-\frac{1}{2}\cos 2\tau\right)\Big|_{\tau=0}^{\tau=t} -$$

$$-\frac{1}{2}\cos t\left(\tau - \frac{1}{2}\sin 2\tau\right)\Big|_{\tau=0}^{\tau=t} = -\frac{1}{4}\sin t\cdot(\cos 2t - 1) - \frac{1}{2}\cos t\left(t - \frac{1}{2}\sin 2t\right) =$$

$$= \frac{1}{2}\left[-\frac{1}{2}\sin t\,(\cos 2t - 1) - \cos t\,(t - \frac{1}{2}\sin 2t)\right] =$$

$$= \frac{1}{2}\left[\underbrace{-\frac{1}{2}\sin t\cdot\cos 2t} + \frac{1}{2}\sin t - t\cos t + \underbrace{\frac{1}{2}\cos t\cdot\sin 2t}\right] \text{(=)}$$

$$\sim = \frac{1}{2}\cos t\cdot\sin 2t - \frac{1}{2}\sin t\cdot\cos 2t = \frac{1}{2}\left(\sin 2t\cdot\cos t - \cos 2t\cdot\sin t\right) \overset{(1)}{=} \frac{1}{2}\sin(2t-t) =$$

$$= \frac{1}{2}\sin t$$

$$\text{(=)}\ \frac{1}{2}\left[\frac{1}{2}\sin t + \frac{1}{2}\sin t - t\cos t\right] = \frac{1}{2}\left[\sin t - t\cos t\right]$$