Lecture 42

System Analysis and Duhamel's Principle

Consider

$$ax'' + bx' + cx = f(t), \qquad a, b, c : \text{const} \qquad (1)$$

$f(t)$: input function

$x(t)$: output / response function

Let $x(0) = x'(0) = 0$. Let $X(s) = \mathcal{L}\{x(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$. Then applying Laplace transform to both sides of (1) we get

$$a\left(s^2 X(s) - s\overset{0}{\cancel{x(0)}} - \overset{0}{\cancel{x'(0)}}\right) + b\left(s X(s) - \overset{0}{\cancel{x(0)}}\right) + c X(s) = F(s)$$

$$\Rightarrow \quad (as^2 + bs + c) X(s) = F(s)$$

$$\Rightarrow \quad X(s) = \frac{F(s)}{as^2 + bs + c} = \underbrace{\frac{1}{as^2 + bs + c}}_{\equiv W(s)} \cdot F(s) = W(s) \cdot F(s) : \text{ Laplace transform of response}$$

$W(s) = \dfrac{1}{as^2 + bs + c}$: transfer function of system

$w(t) = \mathcal{L}^{-1}\{W(s)\}$: weight function of system or impulse response function (more - next page)

$X(s) = W(s)\cdot F(s) \quad \Rightarrow$

$$\boxed{x(t) = w(t) * f(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau} :$$

Duhamel's principle

Note

$$W(s) = \frac{1}{as^2+bs+c} = \frac{1}{as^2+bs+c}\cdot \underset{\mathcal{L}\{\delta(t)\}}{\overset{\parallel}{1}} = W(s)\cdot \mathcal{L}\{\delta(t)\} :$$

Laplace transform of response due to input $f(t)=\delta(t)$

$$\therefore \quad w(t) = w(t) * \delta(t) = \int_0^t w(\tau)\, \delta(t-\tau)\, d\tau = w(t)$$

$\therefore$ weight function $w(t)$ is response of system to delta function $\delta(t) \Rightarrow$ $w(t)$ is called the <u>impulse response</u> function

Let $f(t) = u(t)$: unit step function,

$x(t) = h(t)$: response to $u(t)$ (it is easier to measure than response to $\delta(t)$)

$X(s) = W(s)\cdot F(s)$ or $x(t) = w(t) * f(t)$

$\therefore \quad H(s) = \mathcal{L}\{h(t)\} = W(s)\cdot \mathcal{L}\{u(t)\}$ or $h(t) = w(t) * u(t)$

but $\mathcal{L}\{u(t)\} = \frac{1}{s} \Rightarrow H(s) = \overline{W}(s) \cdot \frac{1}{s}$

and $h(t) = w(t) * u(t) = \int_0^t w(\tau)\, u(t-\tau)\, d\tau$ ⊜

$0 \leq \tau \leq t \Rightarrow t-\tau \geq 0$

$$u(t-\tau) = \begin{cases} 1, & t-\tau \geq 0 \\ 0, & t-\tau < 0 \end{cases}$$

⊜ $\int_0^t w(\tau) \cdot 1\, d\tau = \int_0^t w(\tau)\, d\tau \Rightarrow \boxed{h(t) = \int_0^t w(\tau)\, d\tau}$

$\Rightarrow h'(t) = \frac{d}{dt} \int_0^t w(\tau)\, d\tau = w(t) \Rightarrow$

$\boxed{w(t) = h'(t)}$ weight function or unit impulse response function is the derivative of unit step function response

Then response $x(t)$ (Duhamel's principle) can be written as

$$x(t) = w(t) * f(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau = \int_0^t h'(t)\, f(t-\tau)\, d\tau$$

Final Review

#6
S 7.5

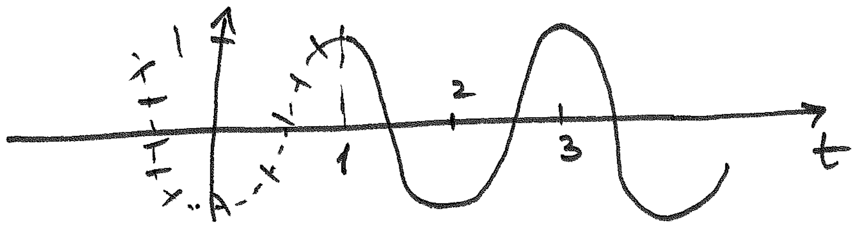
$$F(s) = \frac{se^{-s}}{s^2+\pi^2} = \underset{\substack{a=1 \\ t \to t-1}}{e^{-s}} \cdot \underbrace{\frac{s}{s^2+\pi^2}}_{\mathcal{L}\{\cos \pi t\}}$$

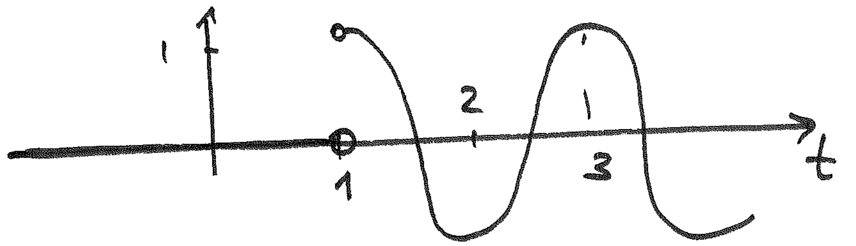
$$\mathcal{L}\{u(t-a) f(t-a)\} = e^{-as} F(s)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos \underbrace{\pi(t-1)}_{\pi t - \pi} \cdot u(t-1) = \Big[\cos \pi t \cdot \underbrace{\cos(\pi)}_{-1} + \sin \pi t \cdot \overset{0}{\cancel{\sin(\pi)}}\Big] u(t-1) \boxed{=}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

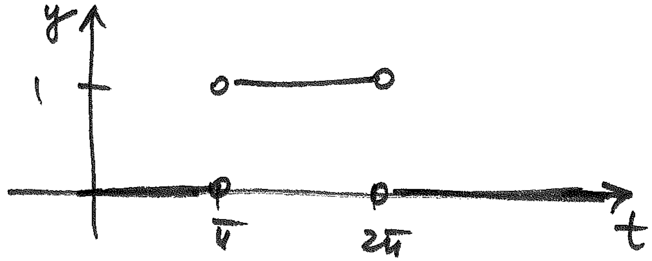
$$\boxed{=} - \cos \pi t \cdot u(t-1)$$





#16
S7.5

$$f(t) = \begin{cases} 0, & t < \pi \\ \sin 2t, & \pi \le t \le 2\pi \\ 0, & t > 2\pi \end{cases}$$

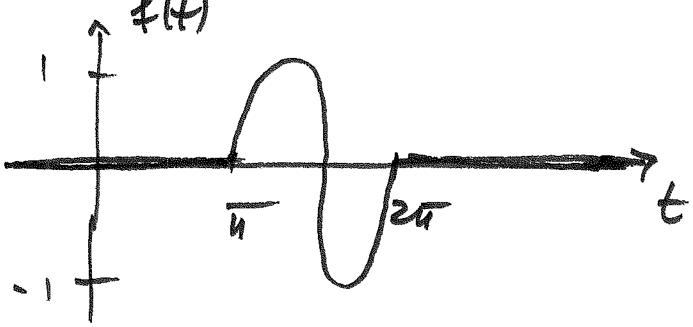


$u(t-\pi) - u(t-2\pi)$

Find $\mathcal{L}\{f(t)\} = F(s)$

$\therefore\ f(t) = \left[u(t-\pi) - u(t-2\pi)\right] \cdot \sin 2t =$

$= u(t-\pi)\sin 2t - u(t-2\pi)\sin 2t \ \boxed{\equiv}$



Shift-Chop Thm:

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$\sin 2t \overset{?}{=} \sin 2(t-\pi) = \sin(2t - 2\pi) = \sin 2t$ ✓

by periodicity

$\sin 2t \overset{?}{=} \sin 2(t-2\pi) = \sin(2t - 4\pi) = \sin 2t$ ✓

by periodicity

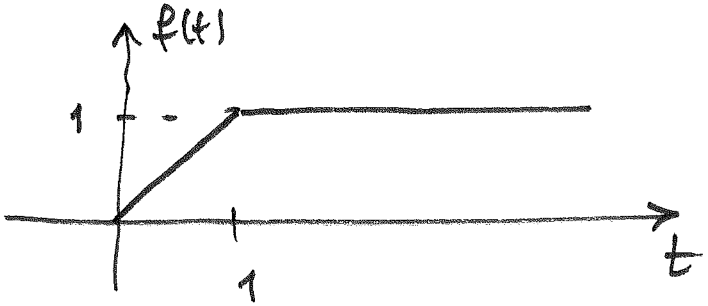
☰ $u(t-\pi)\sin 2(t-\pi) - u(t-2\pi)\sin 2(t-2\pi) = f(t)$

$\underset{a=\pi}{}\qquad\qquad \underset{a=2\pi}{}$

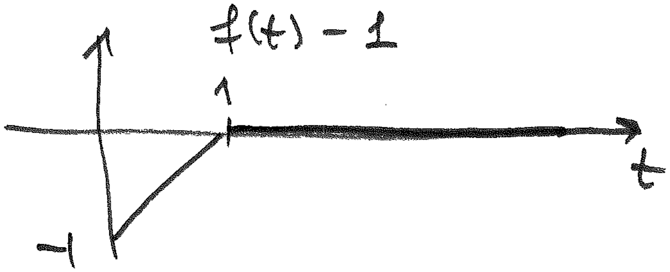
$$\therefore F(s) = \underbrace{\frac{2}{s^2+2^2}}_{\mathcal{L}\{\sin 2t\}} \cdot e^{-\pi s} - \frac{2}{s^2+2^2}\cdot e^{-2\pi s} = \frac{2}{s^2+4}\left(e^{-\pi s} - e^{-2\pi s}\right)$$

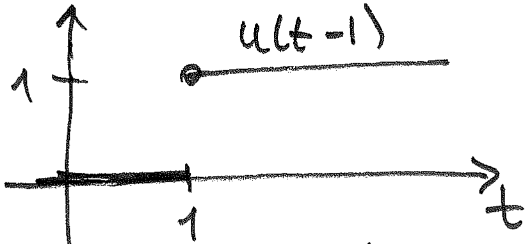
#20
S 7.5

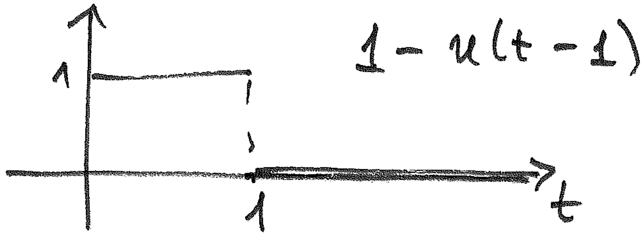
$$f(t) = \begin{cases} t, & t \le 1 \\ 1, & t > 1 \end{cases}$$



$$f(t) - 1 = \begin{cases} t-1, & t \le 1 \\ 0, & t > 1 \end{cases}$$







$\Downarrow$

$$f(t) - 1 = \left[1 - u(t-1)\right]\cdot(t-1)$$

Then

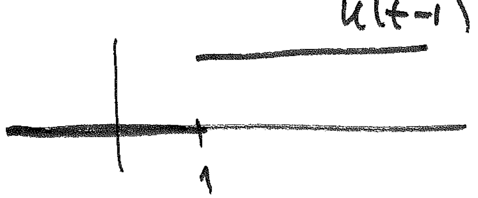
$$f(t) = [1 - u(t-1)](t-1) + 1 = t \cancel{-1} - u(t-1)(t-1) + \cancel{1}$$

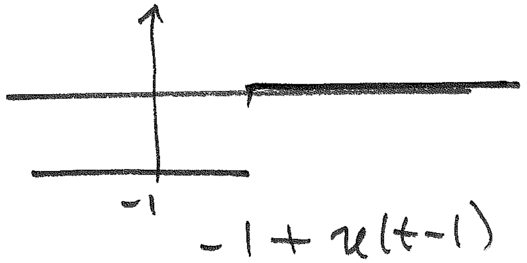
$$\Rightarrow f(t) = t - \underset{a=1}{(t-1)}\, u(t-1)$$

$$\therefore \quad F(s) = \frac{1}{s^2} - \frac{1}{s^2} e^{-s} = \frac{1}{s^2}(1 - e^{-s})$$

$$\left\{ \begin{aligned} &\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \\ &\mathcal{L}\{u(t-a) f(t-a)\} = e^{-as} F(s) \end{aligned} \right.$$

$u(t-1) - 1$



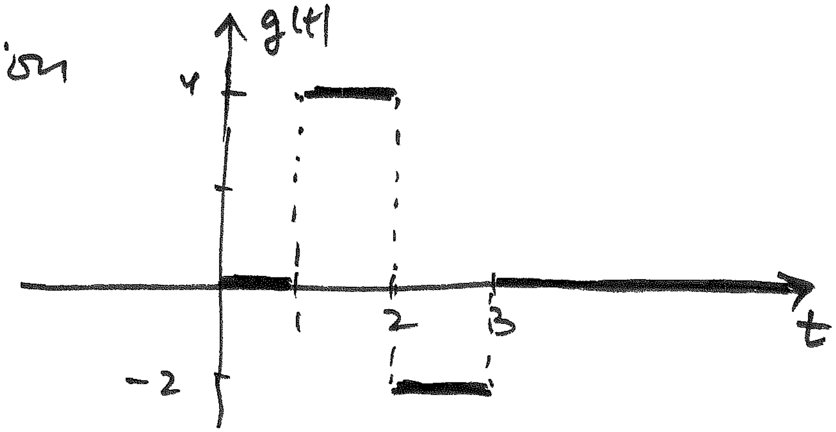


Ex Solve and graph the solution

$$y' + 2y = g(t), \quad y(0) = 6$$

where

$$g(t) = \begin{cases} 0, & 0 < t < 1 \\ 4, & 1 < t < 2 \\ -2, & 2 < t < 3 \\ 0, & t > 3 \end{cases}$$



From the graph: $g(t) = 4u(t-1) - 6u(t-2) + 2u(t-3)$

$$\mathcal{L}\{g(t)\} = G(s) = 4\frac{e^{-s}}{s} - 6\frac{e^{-2s}}{s} + 2\frac{e^{-3s}}{s} \qquad \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Applying Laplace transform to both sides of DE, we get

$$sY(s) - \underset{6}{\underbrace{y(0)}} + 2Y(s) = 4\frac{e^{-s}}{s} - 6\frac{e^{-2s}}{s} + 2\frac{e^{-3s}}{s}$$

$$(s+2)Y(s) = 6 + 4\frac{e^{-s}}{s} - 6\frac{e^{-2s}}{s} + 2\frac{e^{-3s}}{s}$$

$$Y(s) = \frac{6}{s+2} + e^{-s}\frac{4}{s(s+2)} + e^{-2s}\frac{(-6)}{s(s+2)} + e^{-3s}\frac{2}{s(s+2)}$$

$$\frac{4}{s(s+2)} = \frac{2}{s} - \frac{2}{s+2} \qquad \frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2}$$

$$-\frac{6}{s(s+2)} = \frac{-3}{s} + \frac{3}{s+2}$$

$$Y(s) = \frac{6}{s+2} + \left(\frac{2}{s} - \frac{2}{s+2}\right)\underset{a=1}{e^{-s}} + \left(-\frac{3}{s} + \frac{3}{s+2}\right)\underset{a=2}{e^{-2s}} +$$

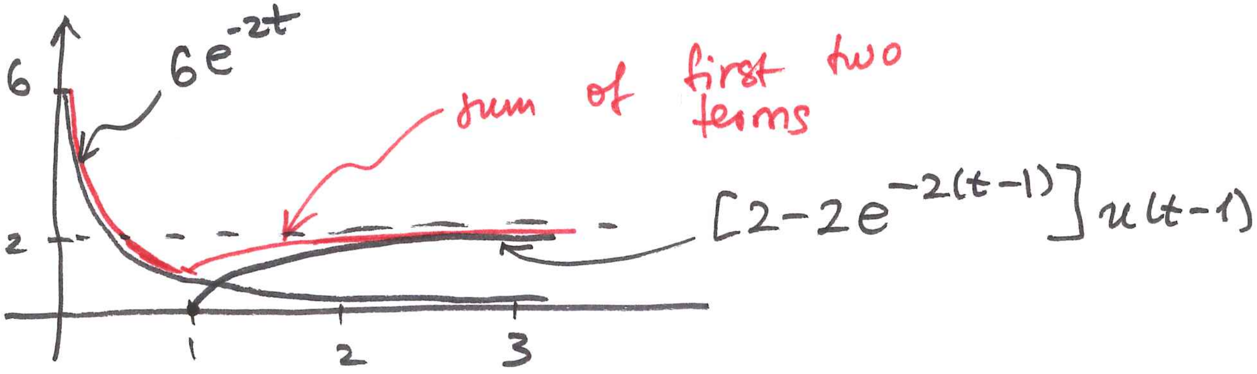
$$+ \left(\frac{1}{s} - \frac{1}{s+2}\right)\underset{a=3}{e^{-3s}}$$

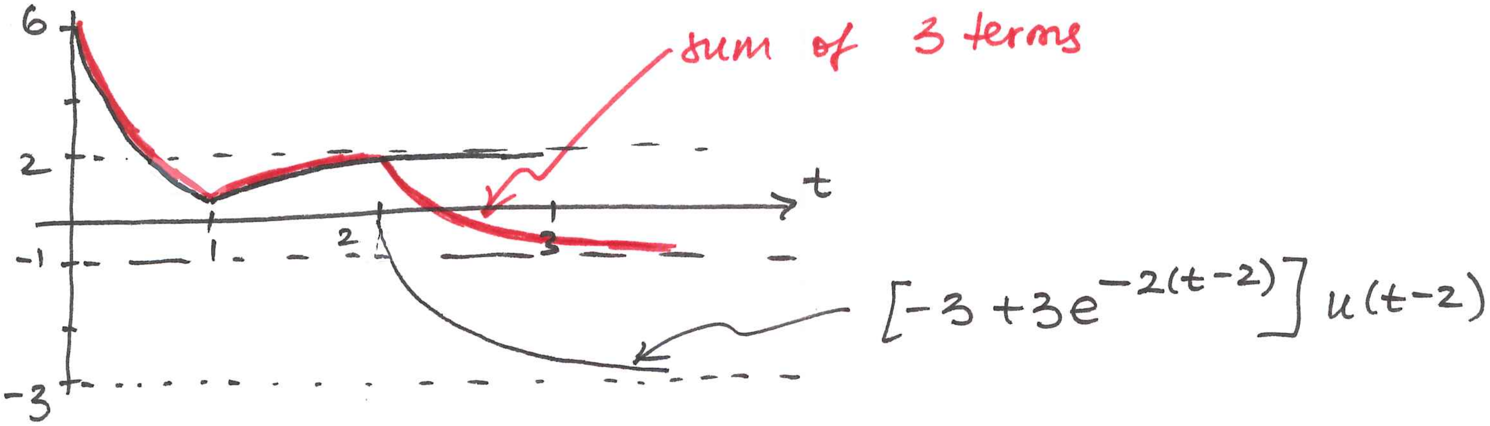
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

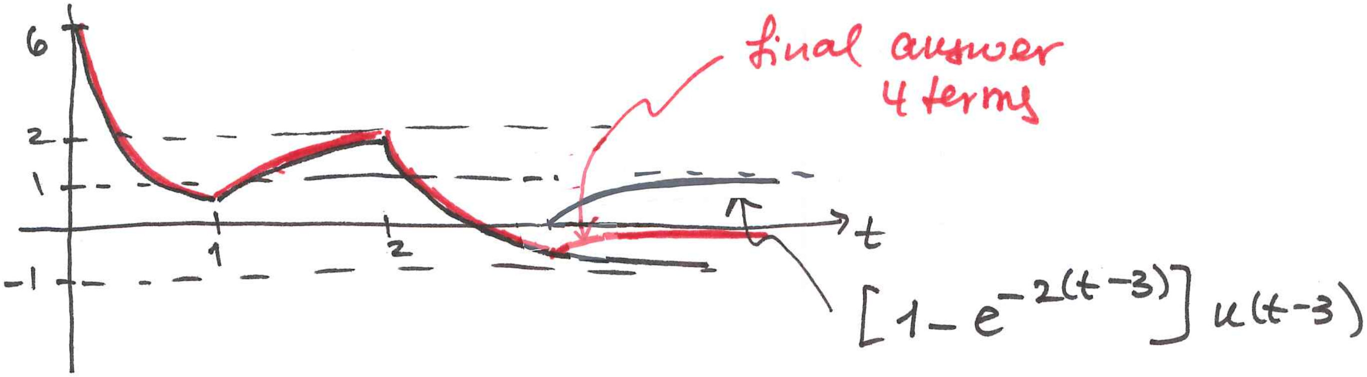
$$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}F(s)$$

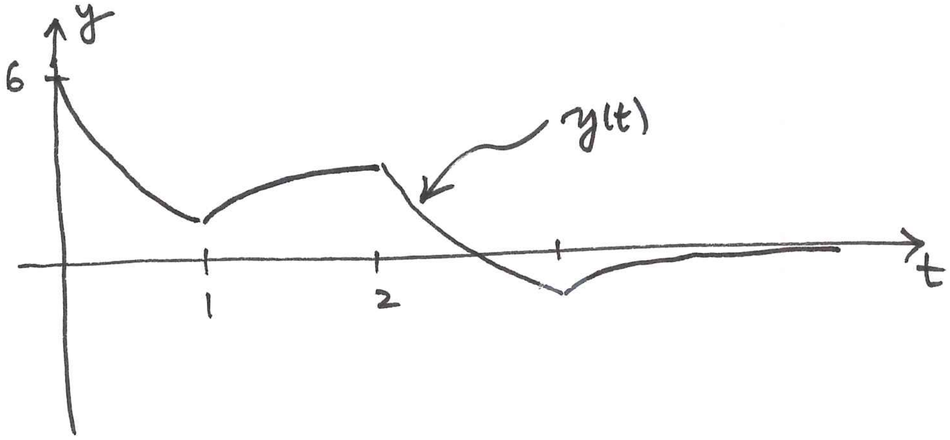
$$y(t) = 6e^{-2t} + \left[2 - 2e^{-2(t-1)}\right]u(t-1) +$$

$$+ \left[-3 + 3e^{-2(t-2)}\right]u(t-2) + \left[1 - e^{-2(t-3)}\right]u(t-3)$$




6
2
-1
-3
1
2
3
t
sum of 3 terms
$[-3+3e^{-2(t-2)}]\,u(t-2)$


6
2
1
-1
1
2
t
final answer
4 terms
$[1-e^{-2(t-3)}]\,u(t-3)$


y
6
1
2
t
y(t)

Math 310: Final Review (More problems)

#21
S1.4

$$2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \quad y(5)=2$$

$$\frac{dy}{dx} = \frac{x}{2y\sqrt{x^2-16}} = \underbrace{\frac{1}{y}}_{f^{n} \text{ of } y} \cdot \underbrace{\frac{x}{2\sqrt{x^2-16}}}_{f^{n} \text{ of } x}$$

: separable ODE (1st order nonlinear)

$$y\,dy = \frac{x\,dx}{2\sqrt{x^2-16}}$$

$$\int y\,dy = \int \frac{x\,dx}{2\sqrt{x^2-16}}$$

$$\frac{y^2}{2} = \frac{1}{2}\sqrt{x^2-16} + C \quad \Big/ \cdot 2$$

$$\int \frac{x\,dx}{2\sqrt{x^2-16}} = \left| \begin{array}{l} u = x^2-16 \\ du = 2x\,dx \end{array} \right| = \int \frac{du}{2\cdot 2\sqrt{u}} = \frac{1}{2}\sqrt{u} + C$$

$$= \frac{1}{2}\sqrt{x^2-16} + C$$

$$y^2 = \sqrt{x^2-16} + \tilde{C}$$

$$y(5)=2 \Rightarrow 2^2 = \sqrt{5^2-16} + \tilde{C}$$

$$4 = 3 + \tilde{C} \Rightarrow \tilde{C} = 1$$

$\Rightarrow \; y^2 = \sqrt{x^2-16}+1$ $\qquad y(5)=2$

$y = \pm\sqrt{\sqrt{x^2-16}+1}$ $\qquad$ choose "+" solution

$\therefore \; \boxed{y = \sqrt{\sqrt{x^2-16}+1}}$

9
S1.5

$xy' - y = x, \quad y(1) = 7$

1st order linear DE

Multiply both sides of DE by $\frac{1}{x}$.

$y' + P(x)y = Q(x)$

$\rho(x) = e^{\int P(x)dx}$: integrating factor

$\rho y = \int \rho Q\,dx + C$

$$y' \underbrace{-\frac{1}{x}}_{P} y = \underbrace{1}_{Q}$$

$$\rho(x) = e^{\int P(x)dx} = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$\rho y = \int \rho Q\,dx + C$ $\qquad e^{\ln a} = a$

$\frac{1}{x}\cdot y = \int \frac{1}{x}\, 1\, dx + C$ $\qquad a\ln b = \ln b^a$

$\frac{1}{x} y = \ln x + C$

IC: $y(1) = 7 \Rightarrow \frac{1}{1}\cdot 7 = \cancelto{0}{\ln 1} + C \Rightarrow \boxed{C = 7}$

$\frac{1}{x} y = \ln x + 7 \Rightarrow \boxed{y = x(\ln x + 7)}$

Method of undetermined coefficients

#48 S 3.5

$y'' - 2y' - 8y = 3e^{-2x}$ (1)

$(D^2 - 2D - 8)y = 3e^{-2x}$

$-2, 4$ $\quad -2 \quad A(D) = D+2$

$(D+2)(D-4)y = 3e^{-2x}$

Higher order DE is

$[(D+2)(D-4)](D+2)y = 0$

$-2, 4; -2$

$y(x) = C_1 e^{-2x} + C_2 e^{4x} + K_1 x e^{-2x}$

$y_q = K_1 x e^{-2x}$: candidate for particular solution

Substitute y_q into (1) to find K_1

(−8) $y_q = K_1 x e^{-2x}$

(−2) $y_q' = K_1 e^{-2x} + K_1 x(-2) e^{-2x} = K_1 e^{-2x} - 2K_1 x e^{-2x}$

$y_q'' = -2K_1 e^{-2x} - 2K_1 e^{-2x} + 4K_1 x e^{-2x}$

① $y_g'' = -4K_1 e^{-2x} + 4K_1 x e^{-2x}$

$$\left(-8K_1 - 2(-2)K_1 + 4K_1\right) x e^{-2x} + (-2K_1 - 4K_1) e^{-2x} = 3e^{-2x}$$

$\{e^{-2x}, x e^{-2x}\}$: linear independent

$x e^{-2x}$: $-8K_1 + 4K_1 + 4K_1 = 0$ $\qquad -8K_1 + 8K_1 = 0$ ✓

e^{-2x} : $-6K_1 = 3 \Rightarrow \boxed{K_1 = -\frac{1}{2}}$

$\therefore \boxed{y_p(x) = -\frac{1}{2} x e^{-2x}}$

General solution is

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{4x} - \frac{1}{2} x e^{-2x}}$$

Variation of parameters

(#48 S3.5) $y'' - 2y' - 8y = 3e^{-2x}$

$y'' - 2y' - 8y = 0$: associated homogeneous eq^{n}

$(D^2 - 2D - 8)y = 0$

$(D+2)(D-4)y = 0 \Rightarrow \boxed{y_c = C_1 e^{-2x} + C_2 e^{4x}}$

$-2, 4$

Assume

$$y_p(x) = A_1(x)e^{-2x} + A_2(x)e^{4x}$$

To find A_1, A_2, we solve for A_1', A_2' the following system of equations

$$\begin{cases} y_1 A_1' + y_2 A_2' = 0 \\ y_1' A_1' + y_2' A_2' = \dfrac{R(x)}{a_2(x)} \end{cases}$$

where y_1, y_2 are linearly independent solutions of the associated homogeneous DE, i.e.

$$y_1 = e^{-2x}, \quad y_2 = e^{4x}$$

$R(x) = 3e^{-2x}, \quad a_2 = 1$

$$\begin{pmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 3e^{-2x}/1 \end{pmatrix}$$

Solve using Cramer's rule

$$\Delta = \begin{vmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{vmatrix} = 4e^{-2x}e^{4x} + 2e^{-2x}e^{4x}$$

$$= 4e^{2x} + 2e^{2x} = 6e^{2x}$$

$$\Delta_1 = \begin{vmatrix} 0 & e^{4x} \\ 3e^{-2x} & 4e^{4x} \end{vmatrix} = -3e^{2x}$$

$$\Delta_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 3e^{-2x} \end{vmatrix} = 3e^{-4x}$$

Then

$$A_1' = \frac{\Delta_1}{\Delta} = \frac{-3e^{2x}}{6e^{2x}} = -\frac{1}{2}$$

$$A_1(x) = \int \left(-\frac{1}{2}\right) dx = \boxed{-\frac{1}{2}x = A_1(x)}$$

$$A_2' = \frac{\Delta_2}{\Delta} = \frac{3e^{-4x}}{6e^{2x}} = \frac{1}{2}e^{-6x}$$

$$A_{2(x)} = \int \frac{1}{2}e^{-6x}\,dx = \boxed{-\frac{1}{12}e^{-6x} = A_2(x)}$$

Hence,

$$y_p(x) = A_1 \cdot y_1 + A_2 \cdot y_2 = -\frac{1}{2}x \cdot e^{-2x} - \frac{1}{12}\underbrace{e^{-6x} \cdot e^{4x}}_{e^{-2x}}$$

$$= -\frac{1}{2}x e^{-2x} - \frac{1}{12}e^{-2x}$$

Then, the general solution is

$$y(x) = \underbrace{C_1 e^{-2x} + C_2 e^{4x}}_{y_c} \underbrace{-\frac{1}{2}x e^{-2x} - \frac{1}{12}e^{-2x}}_{y_p} \ ⊜$$

$$\textcircled{=}\ \tilde{C}_1 e^{-2x} + C_2 e^{4x} \overbrace{- \frac{1}{2} x e^{-2t}}^{= y_p}$$

$\tilde{C}_1 = C_1 - \frac{1}{12}$: arbitrary constant

arbitrary constant

#21
S7.4

$$f(t) = \frac{e^{3t} - 1}{t}$$

Thm: integration of transforms

$$\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(\sigma)\, d\sigma \qquad |f(t)| \le M e^{\alpha t} \text{ as } t \to \infty$$

$$\mathcal{L}\left\{ \frac{e^{3t} - 1}{t} \right\} = \int_s^\infty \mathcal{L}\{e^{3t} - 1\}\, d\sigma \ \textcircled{=}$$

$$\mathcal{L}\{e^{3t} - 1\} = \frac{1}{s-3} - \frac{1}{s}$$

$$\textcircled{=} \int_s^\infty \left(\frac{1}{\sigma - 3} - \frac{1}{\sigma} \right) d\sigma = \left(\ln|\sigma - 3| - \ln|\sigma| \right) \Big|_s^\infty =$$

$$= \ln\left| \frac{\sigma - 3}{\sigma} \right| \Bigg|_s^\infty = \overset{0}{\cancel{\ln 1}} - \ln\left(\frac{s-3}{s} \right) = \underset{s > 3}{\boxed{\ln \frac{s}{s-3}}}$$

$$\lim_{\sigma \to \infty} \frac{\sigma - 3}{\sigma} = \frac{1}{1} = 1 \Rightarrow \ln 1 = 0$$

Find inverse Laplace transform.

#13
S7.4

$$F(s) = \frac{s}{(s-3)(s^2+1)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+1} =$$

$$= \frac{A(s^2+1) + (Bs+C)(s-3)}{(s-3)(s^2+1)}$$

$$\Rightarrow s = A(s^2+1) + (Bs+C)(s-3)$$

s^2: $0 = A + B \Rightarrow B = -A$

s^1: $1 = -3B + C$

s^0: $0 = A - 3C \quad C = \frac{A}{3}$

$$1 = -3(-A) + \frac{A}{3}$$

$$1 = \left(3 + \frac{1}{3}\right)A \Rightarrow 1 = \frac{10}{3} A \Rightarrow A = \frac{3}{10} = 0.3$$

$B = -A = -0.3$, $\quad C = \frac{A}{3} = 0.1$

$$\therefore F(s) = \frac{0.3}{s-3} + \frac{-0.3s + 0.1}{s^2+1} =$$

$$= 0.3 \frac{1}{s-3} - 0.3 \frac{s}{s^2+1} + 0.1 \frac{1}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 0.3e^{3t} - 0.3\cos t + 0.1 \sin t$$

$$F(s) = \frac{s}{(s-3)(s^2+1)} = \frac{1}{s-3} \cdot \frac{s}{s^2+1} = \mathcal{L}\{e^{3t}\} \cdot \mathcal{L}\{\cos t\}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{3t} * \cos t =$$

$$= \int_0^t e^{3\tau} \cos(t-\tau)\, d\tau = \int_0^t \underbrace{e^{3(t-\tau)}}_{e^{3t}e^{-3\tau}} \cos\tau\, d\tau =$$

$$= e^{3t} \underbrace{\int_0^t e^{-3\tau} \cos\tau\, d\tau}_{I} \ :$$ twice by parts to get equation for I

Thus: $f*g = g*f$