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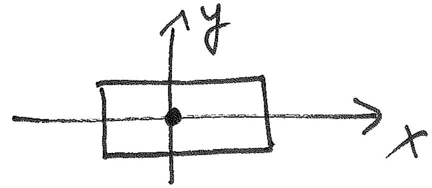
Lecture 4

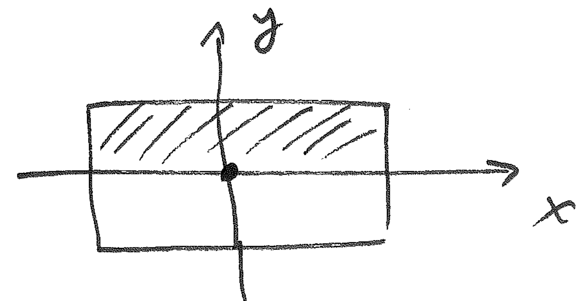


$\underline{\underline{\text{Ex}}}$ $y' = \underbrace{\frac{1}{x}}_{f(x,y)}$, $y(0)=0$

but $\frac{1}{x}$ is <u>not</u> continuous at $x=0$ (where IC is given)

$\Rightarrow$ Thm cannot guarantee existence of a solution.

In fact, we know that there is <u>NO</u> solution.



$\underline{\underline{\text{Ex}}}$ $y' = \underbrace{2\sqrt{y}}_{f(x,y)}$, $y(0)=0$ $\quad x \geq 0$

$f = 2\sqrt{y}$ is cts for $x \geq 0$, $y \geq 0$. $\Rightarrow$ solution exists

$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$ is not cts at $y=0$ where IC is given

$$\frac{d}{dy}\left(y^{\frac{1}{2}}\right) = \frac{1}{2} y^{\frac{1}{2}-1} = \frac{1}{2\sqrt{y}}$$

Since $\frac{\partial f}{\partial y}$ is not cts at $y=0$ $\Rightarrow$ Thm cannot guarantee that solution is unique.

In fact, we know that there are two solutions:

$$y_1 = x^2, \quad y_2 \equiv 0.$$

1.4 Separable Equations and Applications

Def a first-order DE

$$\frac{dy}{dx} = H(x,y)$$

is **separable** if $H(x,y)$ can be written as

$$H(x,y) = g(x)\,h(y) = \frac{g(x)}{f(y)} \qquad \text{where} \quad h(y) = \frac{1}{f(y)}$$

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \quad \Big| \cdot f(y)\,dx$$

$f(y)\,dy = g(x)\,dx$: variables x and y are separated

$$\int f(y)\,dy = \int g(x)\,dx$$

$F(y) = G(x) + C$: a general solution of $\dfrac{dy}{dx} = \dfrac{g(x)}{f(y)}$

implicit solution for y

This method is called <u>the method of separation of variables</u>.

Note

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \quad \Big| \cdot f(y)$$

$$f(y)\frac{dy}{dx} = g(x) \qquad y = y(x)$$

$$\int f(y) \underbrace{\frac{dy}{dx}\, dx}_{dy} = \int g(x)\, dx$$

← change of variables

Ex

$$\frac{dP}{dt} = k P^2 \quad \Big| \cdot \frac{dt}{P^2} \qquad P \neq 0$$

$$\underbrace{\frac{dP}{P^2}}_{f^n \text{ of } P} = \underbrace{k\, dt}_{f^n \text{ of } t}$$

variables are separated

$$\int \frac{dP}{P^2} = \int k\, dt$$

$-\frac{1}{P} = kt + C$: implicit solution for P

$\boxed{P = -\frac{1}{kt + C}}$: explicit solution for P

IC: $P(0) = 2$

$t = 0$: $\underset{\underset{2}{\shortparallel}}{P(0)} = -\frac{1}{k \cdot 0 + C} \Rightarrow 2 = -\frac{1}{C} \Rightarrow C = -\frac{1}{2}$

$\therefore$ $\boxed{P(t) = -\frac{1}{kt - \frac{1}{2}}}$: particular solution

Check if $P \equiv 0$ is another solution.

$\frac{dP}{dt} = kP^2$ $\quad\quad P \equiv 0 \rightarrow P' = 0$

$0 = 0$ ✓ $\Rightarrow P \equiv 0$ satisfies $\frac{dP}{dt} = kP^2$

but $P \equiv 0$ does <u>not</u> satisfy IC $P(0) = 2$

$\therefore$ $P \equiv 0$ is <u>not</u> a solution of IVP $\frac{dP}{dt} = kP^2, \; P(0) = 2$

Ex Solve $\frac{dy}{dx} = -6xy$, $y(0) = 7$

$\frac{dy}{y} = -6x\,dx$, $y \neq 0$

but IC $y(0) = 7 \neq 0$

$\Rightarrow y \equiv 0$ is not a solution of IVP

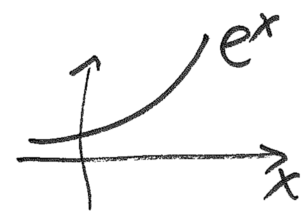
$$\int \frac{dy}{y} = -6\int x\,dx$$

$$\ln|y| = -3x^2 + \tilde{C} \quad \Big| \ \exp$$

$$|y| = e^{-3x^2 + \tilde{C}}$$

$$|y| = \underbrace{e^{-3x^2}}_{>0} \cdot \underbrace{e^{\tilde{C}}}_{>0}$$

$\rightarrow |y| > 0$



$e^{\ln t} = t$

$\ln e^t = t$

$e^{a+b} = e^a \cdot e^b$

$e^{a+b} \neq e^a + e^b$

$\ln(a \cdot b) = \ln a + \ln b$

$\ln \frac{a}{b} = \ln a - \ln b$

or $\boxed{y = Ce^{-3x^2}}$

C can be >0, $=0$, or <0

this also includes solution $y=0$

Note

C can be of different sign or 0, whereas $e^{\tilde{C}} > 0$ always.

IC: $y(0) = 7$

$$\underset{7}{y(0)} = Ce^{-3\cdot 0^2} = C \Rightarrow C = 7$$

$e^0 = 1$

$\therefore$ $\boxed{y(x) = 7e^{-3x^2}}$

<u>Ex</u> $\frac{dy}{dx} = 2\sqrt{y}$, $y(0) = 0$

$x \geq 0$

We showed earlier that this IVP has two solutions

$y_1 = x^2$, $y_2 \equiv 0$

$\frac{dy}{2\sqrt{y}} = dx$, $y \neq 0$

$$\int \frac{dy}{2\sqrt{y}} = \int dx$$

$$\sqrt{y} = x + C$$

$$y = (x+C)^2$$

Check if $y \equiv 0$ is a solution

$$\frac{dy}{dx} = 2\sqrt{y} \qquad y(0) = 0$$

$$0 = 2 \cdot \sqrt{0} \Rightarrow 0 = 0$$

$$y(0) = 0 \checkmark$$

$\therefore$ $y \equiv 0$ is another solution

We can't combine these two solutions in one formula.

the general solution

$$y = \begin{cases} (x+C)^2 & \text{— has an arbitrary const} \\ 0 & \text{— has no arbitrary const} \end{cases}$$

two solutions

$y = (x+C)^2$ is a regular solution

$y \equiv 0$ is called a singular solution

IC $\quad y(0) = 0 \qquad y = (x+C)^2$

at $x = 0 \qquad \underset{0}{\underset{\parallel}{y(0)}} = (0+C)^2 \Rightarrow \boxed{C = 0}$

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$\therefore \quad y(x) = \begin{cases} x^2 \\ 0 \end{cases}$ particular solution

Applications

Natural Growth & Decay

$\frac{dx}{dt} = kx$: separable ODE, $k>0$ or $k<0$

$x \equiv 0 \Rightarrow x' = 0$

$x' = kx$

$0 = k \cdot 0$ ✓

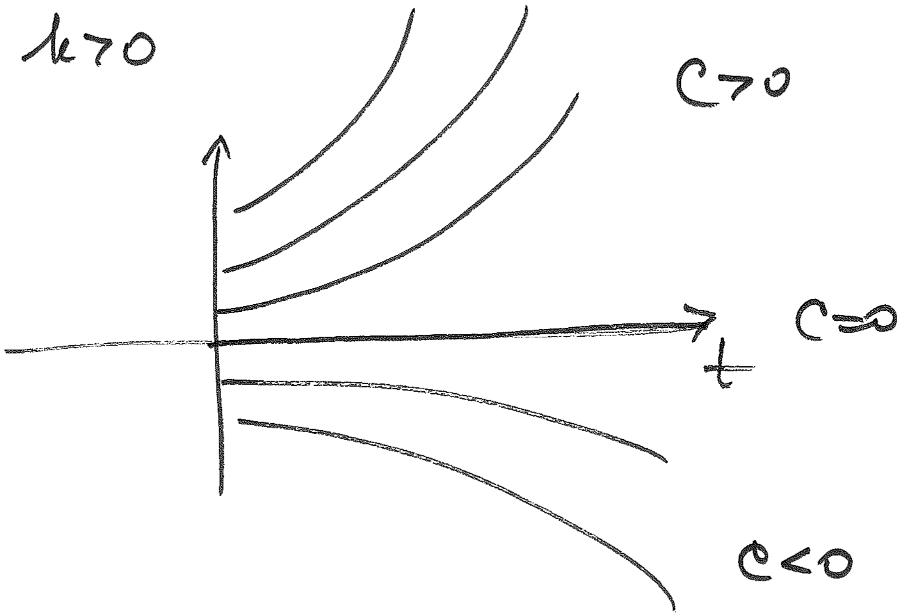
$\therefore x \equiv 0$ is also a sol^{n}

$\frac{dx}{x} = k\,dt$, $x \neq 0$

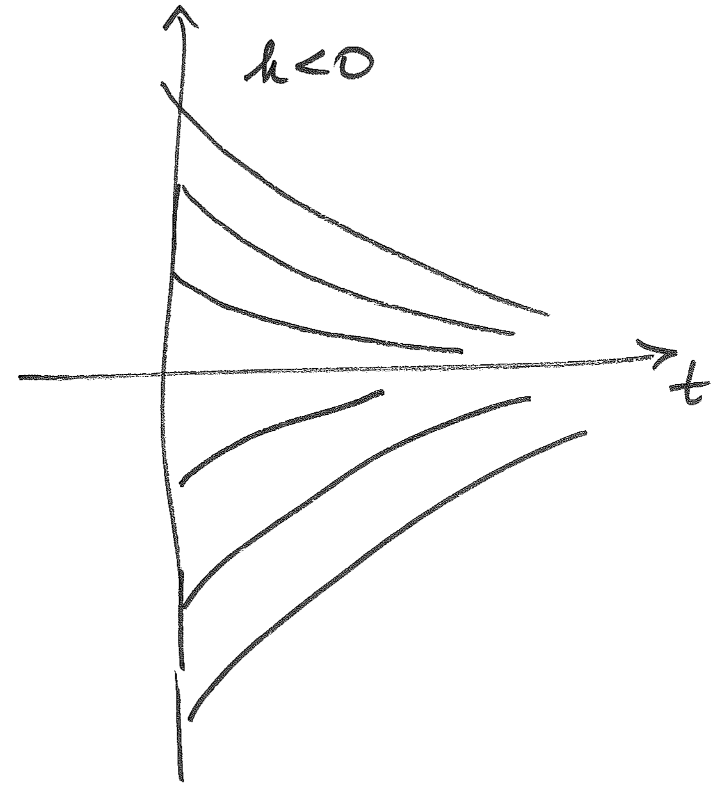
$\int \frac{dx}{x} = \int k\,dt$

$\ln|x| = kt + \tilde{C} \quad | \exp$

$|x| = e^{kt + \tilde{C}}$ or $\boxed{x = Ce^{kt}}$ includes $x>0$, $x \equiv 0$, $x<0$



exponential growth



exponential decay

IC: $x(0) = x_0$

$x(t) = Ce^{kt}$

$t=0$ $\qquad \underset{x_0}{\overset{\|}{x(0)}} = C\underbrace{e^{k\cdot 0}}_{1} \Rightarrow C = x_0 \Rightarrow \boxed{x(t) = x_0 e^{kt}}$