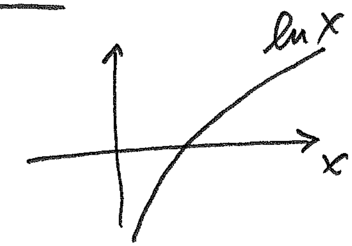
Lecture 4

Existence and Uniqueness of Solutions (§1.3)

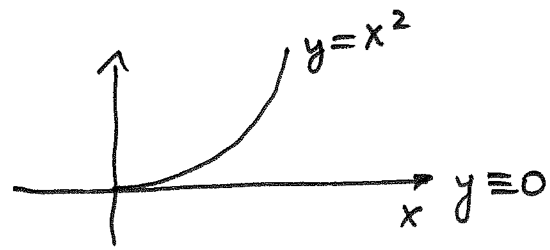
1) $y' = \frac{1}{x}$, $y(0) = 0$



$y(x) = \int \frac{1}{x}\,dx = \ln|x| + C$: not defined at $x = 0$

$\Rightarrow$ this IVP has NO solution

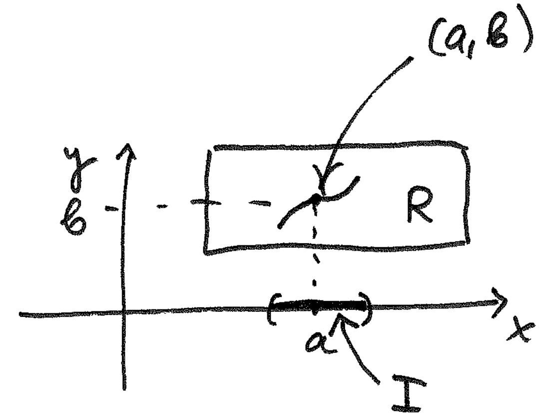
2) $y' = 2\sqrt{y}$, $y(0) = 0$, $x > 0$



$y_1(x) = x^2$, $y_2(x) \equiv 0$

This IVP has more than one solution.

Thm (Existence and Uniqueness of Solutions)

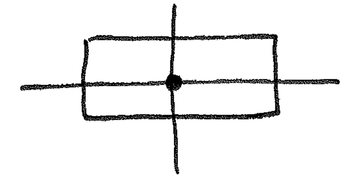


Consider IVP

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b \qquad (1)$$

Suppose that both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on some rectangular domain R that contains pt (a,b).

Then on some interval $\underline{I}$ containing pt a there exists a unique solution $y(x)$ of IVP (1).

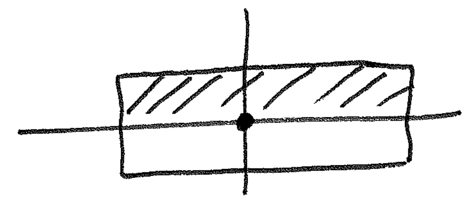


$\underline{\underline{\text{Ex}}}$ $y' = \underbrace{\frac{1}{x}}_{f(x,y)}$, $y(0)=0$

$\Rightarrow$ $f = \frac{1}{x}$ is not continuous at $x=0$

$\Rightarrow$ Thm cannot guarantee existence.

In fact, we know that there is <u>no</u> solution.



$\underline{\underline{\text{Ex}}}$ $y' = \underbrace{2\sqrt{y}}_{f(x,y)}$, $y(0)=0$, $x \geq 0$

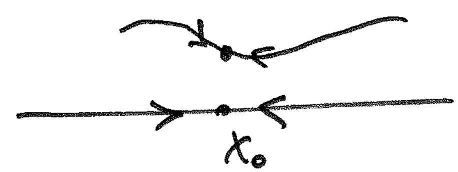
$f(x,y)$ is continuous for $y \geq 0$

Since $f(x,y)$ is continuous for $y \geq 0$, solution exists (by Thm).

$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$: not continuous at $y=0$

$\Rightarrow$ Thm cannot guarantee uniqueness of solution. In fact, we have two solutions: $y_1 = x^2$, $y_2 \equiv 0$.

Recall, $f(x)$ is continuous at $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$



1.4 Separable Equations and Applications

Def a first-order DE

$$\frac{dy}{dx} = H(x,y)$$

is separable if $H(x,y)$ can be written as

$$H(x,y) = g(x)\,h(y) = \frac{g(x)}{f(y)} \quad \text{where} \quad f(y) = \frac{1}{h(y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{g(x)}{f(y)} \quad \Big| \cdot f(y) \qquad\qquad f(y)\,dy = g(x)\,dx$$

$$f(y)\frac{dy}{dx} = g(x) \qquad\qquad y = y(x)$$

$$\int f(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

$$\underbrace{\int f(y)\,dy}_{F(y)} = \underbrace{\int g(x)\,dx}_{G(x)}$$

$F(y) = G(x) + C$: <u>a general solution</u> of $\frac{dy}{dx} = \frac{g(x)}{f(y)}$

implicit solution for y

This method is called <u>separation of variables.</u>

<u>Note</u> $\frac{dy}{dx} = \frac{g(x)}{f(y)} \quad \Big| \cdot f(y)\,dx$

$\underbrace{f(y)\,dy}_{f^{n} \text{ of } y} = \underbrace{g(x)\,dx}_{f^{n} \text{ of } x}$: variables are separated

$\int f(y)\,dy = \int g(x)\,dx$

<u>Ex</u> $\frac{dP}{dt} = k P^2 \quad \Big| \cdot \frac{dt}{P^2} \qquad P = P(t)$

$\frac{dP}{P^2} = k\,dt$, if $P \neq 0$

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$$\int \frac{dP}{P^2} = \int k\,dt$$

$-\frac{1}{P} = kt + C$: implicit solution for P

$P(t) = -\frac{1}{kt+C}$: explicit solution, a general solution

IC: $P(0) = 2$

at $t=0$: $\underbrace{P(0)}_{=2} = -\frac{1}{k\cdot 0 + C} \Rightarrow 2 = -\frac{1}{C} \Rightarrow C = -\frac{1}{2}$

$\therefore \; P(t) = -\frac{1}{kt - \frac{1}{2}}$: particular solution

Verify if $P(t) \equiv 0$ is a solution. $\Rightarrow P' = 0$

$P' = kP^2$

$0 = k\cdot 0^2$ ✓ but $P(0) \neq 2$

$\Rightarrow P \equiv 0$ is not a solution of IVP

Ex Solve $\frac{dy}{dx} = -6xy$, $y(0) = 7$

$y \equiv 0 \Rightarrow y' = 0$

$y' = -6xy$

$0 = -6x \cdot 0$ ✓

$\frac{dy}{y} = -6x\,dx$, $y \neq 0$

$\int \frac{dy}{y} = \int (-6)x\,dx$

$\ln|y| = -3x^2 + \tilde{c}$ $\quad | \exp$

$e^{\ln|y|} = e^{-3x^2 + \tilde{c}}$

$|y| = \underbrace{e^{-3x^2}}_{>0} \cdot \underbrace{e^{\tilde{c}}}_{>0}$

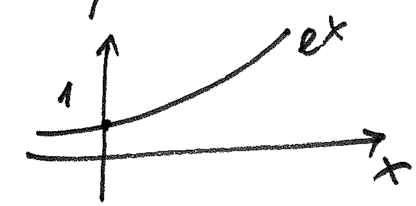
$y(x) = c\underbrace{e^{-3x^2}}_{>0}$ $\quad (*)$

$e^{\ln t} = t$

$\ln e^t = t$

$e^{a+b} = e^a \cdot e^b$

$e^{a+b} \neq e^a + e^b$



By writing $y(x)$ in form $(*)$, we allow $y(x)$ to be >0, <0 or $=0$.

c may be >0, <0 or $=0$. unlike $e^{\tilde{c}} > 0$ always.

Ex $\dfrac{dy}{dx} = 2\sqrt{y}$, $y(0)=0$, $\boxed{x \geq 0}$

separable ODE

We showed earlier that this IVP has two solutions

$y = x^2 \Rightarrow y' = 2x$

$y' \stackrel{?}{=} 2\sqrt{y}$ $\quad \sqrt{y} = \sqrt{x^2} = |x|$

$2x \stackrel{?}{=} 2|x|$ true for $x \geq 0$

$\dfrac{dy}{2\sqrt{y}} = dx$, $y \neq 0$

$\displaystyle\int \frac{dy}{2\sqrt{y}} = \int dx$

$\sqrt{y} = x + C$: implicit solution

$y(x) = (x+C)^2$ $\quad$ IC: $y(0)=0 \Rightarrow \overset{=0}{y(0)} = (0+C)^2 \Rightarrow C = 0 \Rightarrow \boxed{y = x^2}$

Check if $y \equiv 0$ is a solution. $y' = 0$, $\sqrt{y} = 0$

$y' = 2\sqrt{y} \Rightarrow 0 = 2\sqrt{0}$ ✓ $\quad$ IC $y(0) = 0$ ✓

$\Rightarrow y \equiv 0$ is another solution

$\therefore$ $y(x) = \begin{cases} x^2 \\ 0 \end{cases}$ particular solution $\quad$ or $\quad y(x) = \begin{cases} (x+C)^2 \\ 0 \end{cases}$ general solution

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$y(x) = (x+c)^2$: regular solution, has an arbitrary const

$y(x) \equiv 0$: singular solution, has no arbitrary const

Applications

Natural Growth and Decay

$\frac{dx}{dt} = kx$: separable ODE, $k>0$ or $k<0$

$\frac{dx}{x} = k\,dt$, $x \neq 0$ — but $x \equiv 0$ is another solution of $x' = kx$

$\int \frac{dx}{x} = \int k\,dt$

$\ln|x| = kt + \tilde{C}$

$|x| = e^{kt+\tilde{C}}$

or $\boxed{x(t) = Ce^{kt}}$