Lecture 5

Population Model

$P(t)$: # of individuals in a population

Assume:

$$\left.\begin{array}{ll} \beta>0: & \text{constant birth rate} \\ \delta>0: & \text{constant death rate} \end{array}\right\} \begin{array}{l} \text{per population} \\ \text{per unit of time} \end{array}$$

Δt: small interval of time

$\beta P(t)\Delta t$: # of individuals added to $P(t)$ due to birth during time Δt (# of birth)

$\delta P(t)\Delta t$: # of death during Δt

$$\Delta P = \beta P(t)\,\Delta t - \delta P(t)\,\Delta t$$

change in population during Δt

Then $\frac{\Delta P}{\Delta t} = \underbrace{(\beta - \delta)}_{\equiv k} P$ $\qquad k = \beta - \delta$

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = kP \Rightarrow \boxed{\frac{dP}{dt} = kP}$$

: natural growth/decay model, population model

rate of change of P wrt t is proportional to P

Solution: $P(t) = P_0 e^{kt}$

here $P_0 = P(0)$

Radioactive Decay

$N(t)$: # of atoms of a certain radioactive isotope in a sample

Observations: atoms (isotopes) decay by the same fraction.

They decay to other isotopes of the same material or isotopes of other material.

$N(t)$ behaves like a population model with $\beta = 0$,

$\delta > 0$. "const

$$\frac{dN}{dt} = (\cancel{\beta}^{\,0} - \delta) N \Rightarrow \frac{dN}{dt} = -\delta N$$

Typically we write

$$\frac{dN}{dt} = -kN, \qquad k > 0 \qquad k\text{: \underline{decay rate}}$$

For ^{14}C, $k \approx 0.0001216$ t in years.

Def The half-life τ of a radioactive isotope is time required for a half of a sample to decay.

$$\frac{dN}{dt} = -kN \qquad N(0) = N_0 : \text{ initial \# of isotopes}$$

at $t = \tau$ $\quad N(\tau) = \frac{1}{2} N_0$

Solution of $\frac{dN}{dt} = -kN$ is $N(t) = N_0 e^{-kt}$

at $t = \tau$ $\quad \underbrace{N(\tau)}_{= \frac{1}{2} N_0} = N_0 e^{-k\tau}$ $\quad \Rightarrow \quad \frac{1}{2} \cancel{N_0} = \cancel{N_0} e^{-k\tau} \quad \Big| \, \frac{1}{N_0}$

$$\frac{1}{2} = e^{-k\tau} \quad \Big| \, \ln$$

$$\boxed{\ln e^x = x}$$

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$$\ln\frac{1}{2} = \ln e^{-k\tau}$$

$$\ln\frac{1}{2} = -k\tau$$

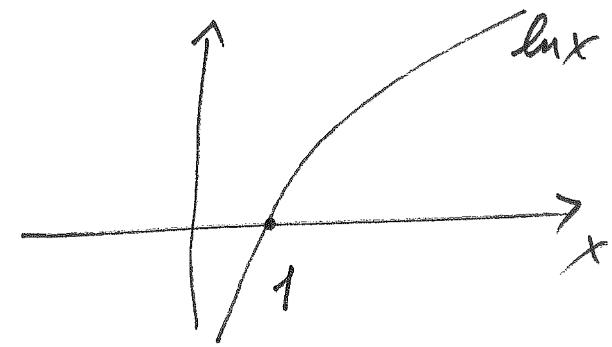
$$\cancel{\ln 1} - \ln 2 = -k\tau$$

0

$$\boxed{\tau = \frac{\ln 2}{k}}$$ <u>half-life time</u>

$$\ln\frac{a}{b} = \ln a - \ln b$$

$$\ln(ab) = \ln a + \ln b$$



For ^{14}C, $\tau = \frac{\ln 2}{k} = \frac{\ln 2}{0.0001216} \approx 5700$ years

Ex Carbon extracted from an ancient skull contained only $\frac{1}{6}$ as much as ^{14}C as carbon extracted from present days bones. How old is the skull? The half-life of ^{14}C is approximately 5700 years.

Solution

$\tau = 5700$ years, $\quad \tau = \frac{\ln 2}{k} \quad \Rightarrow \quad \boxed{k = \frac{\ln 2}{\tau}}$ decay rate

$N(t)$: amount of ^{14}C

$N(0) = N_0$

at $t = t^*$ $\qquad N(t^*) = \frac{1}{6} N_0$

age of skull

$\frac{dN}{dt} = -kN, \quad N(0) = N_0 \quad \Rightarrow \quad N(t) = N_0 e^{-kt}$

at $t = t^*$: $\qquad N(t^*) = N_0 e^{-k \cdot t^*}$, $\quad N(t^*) = \frac{1}{6} N_0$

$$\Rightarrow \frac{1}{6} \cancel{N_0} = \cancel{N_0} e^{-k \cdot t^*}$$

$\frac{1}{6} = e^{-kt^*}$ solve for t^*

$k = \frac{\ln 2}{\tau}$

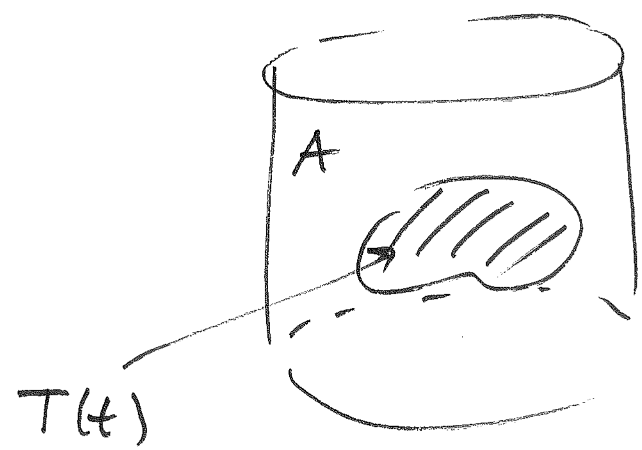
$\ln \frac{1}{6} = \ln e^{-k \cdot t^*}$

$-\ln 6 = -k \cdot t^* \Rightarrow t^* = \frac{\ln 6}{k} = \frac{\ln 6}{\frac{\ln 2}{\tau}} = \frac{\tau \ln 6}{\ln 2} =$

$= \frac{5700 \cdot \ln 6}{\ln 2} \approx 14{,}735$ years

Newton's Law of Cooling/Heating

$T(t)$: temperature of a body immersed in a medium of const temperature A



$$\frac{dT}{dt} = -k(T-A), \qquad k > 0$$

Rate of change of temperature is proportional to the difference of $T(t)$ and temperature A of the surrounding medium.

Solve $\frac{dT}{dt} = -k(T-A)$: separable ODE

$$\frac{dT}{T-A} = -k\,dt \qquad T \neq A$$

$$\ln|T-A| = -kt + \tilde{C} \qquad | \exp$$

$$e^{\ln|T-A|} = e^{-kt+\tilde{C}}$$

$$|T-A| = e^{-kt} \cdot \underbrace{e^{\tilde{C}}}_{>0}$$

$$T - A = C e^{-kt}$$

$$T(t) = A + C e^{-kt}$$

IC $\quad T(0) = T_0$

$$T - A = C e^{-kt}$$

at $t=0$: $\quad T_0 - A = C \underbrace{e^{-k\cdot 0}}_{=1}$

$$\Rightarrow \boxed{C = T_0 - A}$$

$$\boxed{T(t) = A + (T_0 - A)e^{-kt}}$$

solution of $\dfrac{dT}{dt} = -k(T-A)$, $\quad T(0) = T_0$

Ex a pitcher of buttermilk initially at 25°C is cooled on the front porch where temperature is 5°C. Suppose the temperature dropped to 18°C in 10 min. When will it be 10°C?

$T_0 = 25°C$

$A = 5°C$

$T(t)\Big|_{t=10\text{ min}} = 18°C$

Find $t = t^*$:

$T(t)\Big|_{t=t^*} = 10°C$

$$\frac{dT}{dt} = -k(T-A)$$

k is unknown

$$T(t) = A + (T_0 - A)e^{-kt}$$

$$T(t) = 5 + (25-5)e^{-kt} \quad \text{or} \quad T(t) = 5 + 20e^{-kt}$$

Use $T(10) = 18°$ to find k.

at $t = 10$ min

$$\underset{18}{\underset{\shortparallel}{T(10)}} = 5 + 20 \cdot e^{-k \cdot 10} \Rightarrow 13 = \overset{20}{\cancel{1}}e^{-10k}$$

$$-10k = \ln(13/20) \Rightarrow k = -\frac{\ln(13/20)}{10}$$

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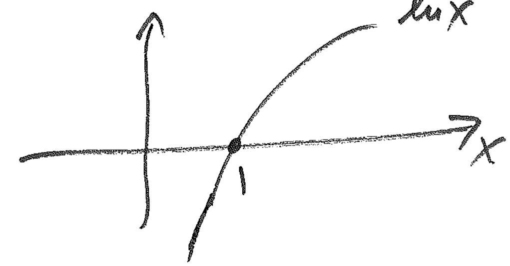
$$\boxed{k = -\frac{\ln \frac{13}{20}}{10}} > 0$$



$\frac{13}{20} < 1$ $\quad \ln \frac{13}{20} < 0$

To find $t = t^*$:

$$T(t^*) = 10$$

$$T(t) = 5 + 20e^{-kt}$$

$$\underbrace{T(t^*)}_{10} = 5 + 20e^{-k \cdot t^*} \Rightarrow 10 = 5 + 20e^{-k \cdot t^*}$$

computed

$$5 = 20e^{-kt^*} \quad \Big| \frac{1}{5}$$

$$e^{-kt^*} = \frac{1}{4}$$

$$-kt^* = \ln \frac{1}{4} = -\ln 4$$

$$t^* = \frac{\ln 4}{k} = -\frac{\ln 4 \cdot 10}{\ln 13/20} \approx 32.18 \text{ min}$$

$$\ln \frac{13}{20} = -\ln \frac{20}{13}$$

$$\Rightarrow \boxed{t^* = \frac{\ln 4 \cdot 10}{\ln 20/13}}$$

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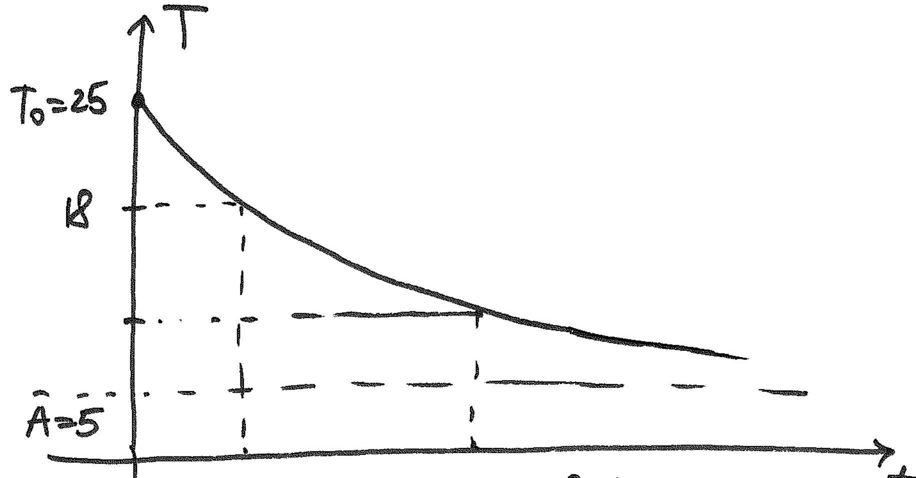

$$t^* = \frac{\ln 4.10}{\ln 20/13} \approx 32.18 \text{ min}$$